On and Asymptotically Sharp Constants in weighted Friedrichs Inequalities for Sobolev Spaces

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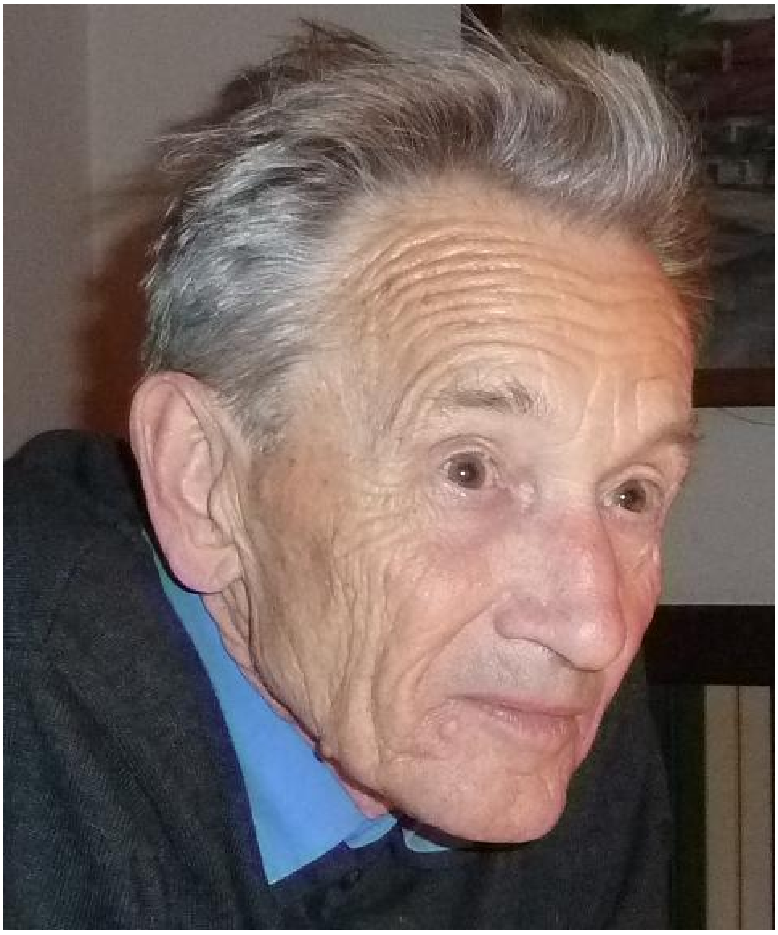
TRIEBEL'S TREE (JENA, 2007)

JENA, OCTOBER 2010



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NOTATIONS AND DEFINITIONS

For $n \in \mathbb{N}$ consider a Sobolev space $\overset{\circ}{W}{}_2^n(-1,1)$, consisting of functions $f : [-1,1] \to \mathbb{R}$ which are absolutely continuous with all derivatives of the order $< n$ and such that

$$f^{(s)}(\pm 1) = 0, \ \ s \in \{0, 1, \ldots, n-1\};$$

$$\|f\|_{\overset{\circ}{W}{}_2^n(-1,1)} := \|f^{(n)}\|_{L_2(dx)} < \infty. \tag{1}$$

POINT-WISE UPPER ESTIMATES

In [K-10] the following assertion has been proved:

Proposition 1. For any $x \in [-1,1]$ the sharp (i.e. the least possible) constant $A_n(x)$ in the inequality

$$f(x) \le A_n(x)\|f^{(n)}\|_{L_2(dx)}, \quad f \in \overset{\circ}{W}{}_2^n(-1,1), \quad (2)$$

is equal to

$$A_n(x) = \frac{(1-x^2)^{n-0.5}}{(n-0.5)^{0.5}2^{n-0.5}(n-1)!} \qquad (3)$$

HISTORICAL COMMENT

In some sense this result is analogous to the brothers Markovs inequality (1887)

$$Q_n^{(k)}(x) \leq M_{n,k}(x) \sup_{t\in[-1,1]} |Q_n(t)|, \quad x \in \mathbb{R} \quad (2^*)$$

for algebraic polynomials $Q_n(t)$.

MARKOV, Andrei Andreevich - 1856 - 1929 - famous Markov chains

MARKOV, Vladimir Andreevich - 1865 - 1890

SIMPLE AND USEFUL EXAMPLE

On the other hand, one can easily obtain

Proposition 2. *For any* $n \in \mathbb{N}$ *introduce a function*

$$y_n(x) := (1 - x^2)^n. \quad (4)$$

Then $y_n(x) \in \overset{\circ}{W}{}_2^n(-1,1)$, *and for all* $x \in (-1,1)$

$$y_n(x) \equiv \frac{(n+0.5)^{0.5}\,(1-x^2)^n}{2^n n!} \|y_n^{(n)}\|_{L_2(dx)} \quad (5)$$

Indeed, $y_n^{(n)}(x)(2^n n!)^{-1}$ is a classical **Legendre** polynomial $P_n(x)$ and $\|P_n\|_{L_2(dx)} = (n+0.5)^{-0.5}$.

Joke by S.B. Stechkin.

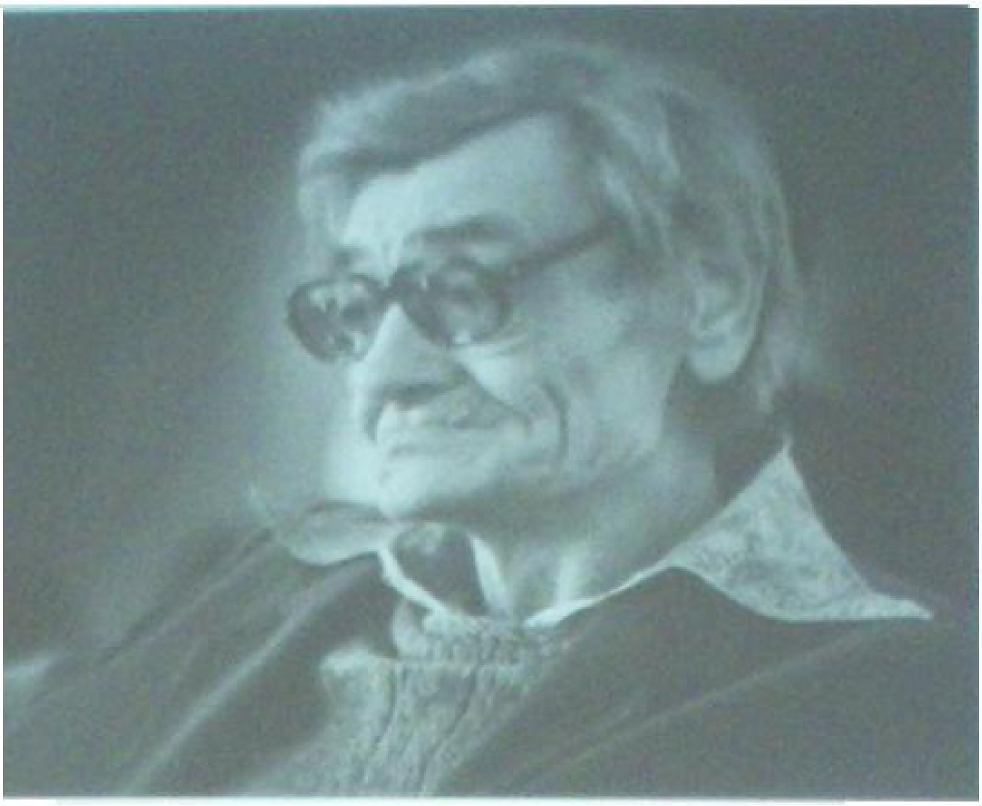
Definition. *The problem is stable (in Stechkin sense) if minor additional efforts cannot essentially improve the result*

or in other words

BE WISE,

DON'T FORGET

TO GENERALIZE!



Sergei Borisovich Stechkin 1920 - 1995

So, further let $0 < q < \infty$, $\mu \not\equiv 0$ be a measure on $(-1,1)$, a quasi-norm (a norm for $q \geq 1$) in $L_q(d\mu)$ is defined as usually:

$$\|f\|_{L_q(d\mu)} := \left(\int_{-1}^{1} |f(t)|^q d\mu(t) \right)^{1/q}. \qquad (6)$$

For $u \in \mathbb{R}$ denote by $\mathcal{I}(u; q, \mu) := \|(1-x^2)^u\|_{L_q(d\mu)}$. This is a positive, non-increasing and convex function of the parameter u. It is clear (cf (5)) that the condition $\mathcal{I}(n; q, \mu) < \infty$ is **necessary** for the embedding $\overset{\circ}{W}{}_2^n(-1,1) \subset L_q(d\mu)$.

On the other hand, from (3) it follows that the condition $\mathcal{I}(n-0.5;q,\mu)<\infty$ is **sufficient** for this embedding.

More precisely, the following assertion holds.

Theorem 1. *Let us introduce a quantity*

$$\mathcal{B}(n;q,\mu) := \| \ \mathrm{id}: \ \overset{\circ}{W}{}_2^n(-1,1) \to L_q(d\mu) \ \| \\ = \sup\{\|f\|_{L_q(d\mu)} : \|f\|_{\overset{\circ}{W}{}_2^n(-1,1)} \le 1\}, \quad (7)$$

and let $\mathcal{I}(u^*;q,\mu)<\infty$ for some $u^*\ge 0$; then for every positive integer $n\ge u^*+0.5$ one has $\underline{\mathcal{B}}(n;q,\mu)\le \ \mathcal{B}(n;q,\mu)\le \ \overline{\mathcal{B}}(n;q,\mu)$, *where*

$$\underline{\mathcal{B}}(n;q,\mu) := \frac{(n+0.5)^{0.5}\ \mathcal{I}(n;q,\mu)}{2^n n!}, \qquad (8)$$

$$\overline{\mathcal{B}}(n;q,\mu) := \frac{\mathcal{I}(n-0.5;q,\mu)}{(n-0.5)^{0.5} 2^{n-0.5} (n-1)!}$$

Remark 1. The ratio of upper and lower estimates

$$\mathcal{R}(n) = \mathcal{R}(n,q,\mu) := \frac{\overline{\mathcal{B}}(n;q,\mu)}{\underline{\mathcal{B}}(n;q,\mu)} \qquad (9)$$

$$= \left(\frac{4n^2}{4n^2-1}\right)^{0.5} \ \frac{\mathcal{I}(n-0.5;q,\mu)}{\mathcal{I}(n;q,\mu)}$$

tends to certain limit (as $n \to +\infty$), which depends only on measure μ.

Theorem 2. *Let us consider a function*

$$\psi_\mu(\delta) := \mu([-\delta, \delta]); \quad 0 \le \delta \le 1, \tag{10}$$

and let $\delta_* = \delta_*(\mu) := \inf\{\delta : \ \psi_\mu(\delta) > 0\}$; *then:*

$$\lim_{n \to +\infty} \mathcal{R}(n, q, \mu) = \mathcal{R}(+\infty, \mu)$$

$$:= (1 - \delta_*^2)^{-0.5}. \tag{11}$$

Proof of Theorem 2 is based on the reasonings used in the solution of the problem **II, 199** in the famous book by G. Polýa and G. Szegö [PS-64].

In the sequel we will consider only the case

$$\mathcal{R}(+\infty, \mu) = 1 \iff \psi_\mu(\delta) > 0 \ \forall \delta > 0. \qquad (12)$$

Evidently, the difference

$$\rho_{\mu,q}(n) := \mathcal{R}(n, q, \mu) - 1 > \frac{1}{4n^2 - 1} \qquad (13)$$

Quantitative estimates show how fast the ratio $\mathcal{R}(n, q, \mu)$ approaches unity.

Theorem 3. *Suppose that* $\psi_\mu(2\delta) < c_0 \psi_\mu(\delta)$ *for some* $c_0 > 0$ *and all* $\delta < \delta_0 < 0.5$. *Then* $\rho_{\mu,q}(n) < c_1/n$.

Remark 2. Under the additional assumption

$$\psi_\mu(\delta) = c\delta^\alpha + o(\delta^\alpha); \ \delta \to +0, c > 0, \alpha > 0, \ (14)$$

one can calculate by standard Laplace-Stieltjes method the main term of asymptotics of $\mathcal{I}(u; q, \mu)$, $u \to +\infty$ and thus of the constants $\mathcal{B}(n; q, \mu)$, namely:

$$\mathcal{B}(n; q, \mu) = \frac{(n + 0.5)^{0.5}}{2^n n!} \left(\frac{c\alpha\Gamma(0.5\alpha)}{(1 + nq)^{0.5\alpha}} \right)^{\frac{1}{q}}$$

$$\times \left(1 + O\left(\frac{1}{n}\right)\right) \qquad (15)$$

Remark 3. The special case $d\mu(x) := dx, q = 2$ was studied by *Böttcher & Widom* (2007), who established the asymptotic formula

$$\mathcal{B}(n; 2, dx) = \frac{1 + O(n^{-0.5})}{2^{0.25}(2n!)^{0.5}} \tag{16}$$

with upper estimate $\asymp n^{0.5}$ times greater than that given by Theorem 1.

REFERENCES

[K-10]. *Kalyabin G.A.* Sharp estimates for derivatives of functions in the Sobolev classes $\mathring{W}_2^r(-1,1)$. Proceedings of the Steklov Institute of Mathematics, **2010**, Volume **269**, p. 143 - 149.

[PS-64]. *Polýa G., Szegö G.* Aufgaben und Lehrsätze aus der Analysis. Erster Band, Springer Verlag, **1964**.

[BW-07] *Böttcher A. , Widom D.* From Töplitz eigenvalues through Green's kernels to higher-order Wirtinger-Sobolev inequalities. // The extended field of operator theory, Oper. Theory Adv. Appl., Basel:Birkhäuser. **2007**. P. 73–87.

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