On Banach space valued Lebesgue spaces

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The Space $L^p(I, X)$

$I = (0, 1)$, $1 \le p < \infty$ and X is a Banach space

$$L^p(I, X) = \left\{ f : I \to X : \|f\|_{L^p(I,X)} = \left(\int_I \|f(x)\|_X^p dx \right)^{\frac{1}{p}} < \infty \right\}$$

(a) For $1 \le p < \infty$, $L^p(I, X)$ is a Banach space.

(b) For $1 \le p < \infty$, $L^p(I, X) \subset L^1(I, X)$

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Dual of $L^p(I, X)$

Definition.

Let X be a Banach space. A function $f : I \to X$ is said to have finite p-variation if

$$V^p(f) = \sup_P \sum_{r=1}^{\alpha} \frac{\|f(x_r) - f(x_{r-1})\|_X^p}{|x_r - x_{r-1}|^{p-1}} < \infty,$$

where sup is taken over all the partitions $P = \{x_0, x_1, ..., x_\alpha\}$ of I.

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$V_0^{p'}(I, X^*)$: set of all $g \in V^{p'}(I, X^*)$ with $g(0) = 0$.

For $1 \leq p < \infty$, $\left[L^p(I, X)\right]^*$ is isometrically isomorphic with $V_0^{p'}(I, X^*)$.

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Weighted version of the space $L^p(I, X)$

$I = (0,1)$, $1 \leq p < \infty$, w is a weight and X is a Banach space

$$L^p_w(I,X) = \left\{ f : I \to X : \|f\|_{L^p_w(I,X)} = \left(\int_I \|f(x)\|^p_X w(x)dx \right)^{\frac{1}{p}} < \infty \right\}$$

- $L^p_w(I,X)$ is a Banach space.
- L^p situation : $f \in L^p_w \iff fw^{1/p} \in L^p$

Not true in $L^p_w(I,X)$ space.

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Another weighted version of the space $L^p(I, X)$

$I = (0,1)$, $1 \le p < \infty$, w is a weight and X is a Banach space

$$\mathscr{L}^p_w(I,X) = \left\{ f : I \to X : \|f\|_{\mathscr{L}^p_w(I,X)} = \left(\int_I \|f(x) w^{1/p}(x)\|_X^p dx \right)^{\frac{1}{p}} < \infty \right\}$$

- $\mathscr{L}^p_w(I,X)$ is a Banach space.
- In general $L^p_w(I,X)$ and $\mathscr{L}^p_w(I,X)$ are different.

Examples.

- $p=2$, $X=L^2(I)$, $f=1$, $w(x)=\frac{1}{x}$. Then

$$f \in L^2_w(I,X) \quad \text{but} \quad f \notin \mathscr{L}^2_w(I,X)$$

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A comparison between $L^p(I, X)$ and $\mathscr{L}_w^p(I, X)$

For $1 < p \leq q < \infty$, $L^q(I, X) \subset L^p(I, X)$ but $\mathscr{L}_w^q(I, X) \subset \mathscr{L}_w^p(I, X)$ is not necessarily true.

Example.

Take $f \equiv 1$, $w(x) = \frac{1}{x^2}$, $X = L^1(I)$. Then

$$f \in \mathscr{L}_w^4(I, X) \qquad \text{but} \qquad f \notin \mathscr{L}_w^2(I, X).$$

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$$\lim_{n \to \infty} \|f_n\|_{V_w^p(I,X)} = 0$$

then the sequence $\{wf_n\}$ converges to zero uniformly on I.

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$V_{0,w}^{p'}(I, X^*)$: set of all $g \in V_w^{p'}(I, X^*)$ with $g(0) = 0$.

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The Sobolev type space $W_w^{1,p}(\Omega, X)$

Definition.

$\Omega \subset \mathbb{R}^n$, α is a multi-index, $1 < p < \infty$ and w is a B_p-weight, i.e., $w^{-1/p-1} \in L^1_{loc}(\Omega)$.

$$W_w^{1,p}(\Omega, X) = \left\{ u \in \mathscr{L}_w^p(\Omega, X) : D^\alpha u \in \mathscr{L}_w^p(\Omega, X) \right\}.$$

$$\|u\|_{W_w^{1,p}(\Omega,X)} := \left(\sum_{|\alpha| \leq 1} \|D^\alpha u\|_{\mathscr{L}_w^p(\Omega,X)} \right)^{1/p}.$$

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Morrey type inequality

Definition.

Let $\Omega \subset \mathbb{R}^n$ be open, $0 \leq \gamma \leq 1$, $1 < p < \infty$ and w be a weight function. A function $u : \Omega \to X$ is said to be weighted Hölder continuous with exponent γ if for $x, y \in \Omega$ and some constant $C > 0$

$$\|u(x)w^{1/p}(x) - u(y)w^{1/p}(y)\|_X \leq C|x - y|^\gamma.$$

Morrey type inequality

Definition.

$C_w^{0,\gamma}(\Omega, X)$: space of all functions which are continuous and weighted Hölder continuous with exponent γ with the norm

$$\|u\|_{C_w^{0,\gamma}(\Omega,X)} = \sup_{x\in U} \|u(x)w^{1/p}(x)\|_X + [u]_{C_w^{0,\gamma}(\Omega)},$$

where

$$[u]_{C_w^{0,\gamma}(\Omega,X)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{\|u(x)w^{1/p}(x) - u(y)w^{1/p}(y)\|_X}{|x-y|^\gamma} \right\}$$

Morrey type inequality

Theorem.

Let X be a Banch algebra, $n < p < \infty, \gamma = 1 - n/p$ and w be a differentiable weight. There exists a constant $C > 0$, depending on p and n such that for all $u \in C^1(\Omega, X)$

$$\begin{aligned}\|u\|_{C_w^{0,\gamma}(\Omega,X)} &\leq C\|u\|_{W_w^{1,p}(\Omega,X)} \\ &\quad + \|u\|_{\mathscr{L}_w^p(\Omega,X)} \left(\int_\Omega \frac{\|D\log w(y)\|_X^{p'}}{|x-y|^{(n-1)p/p-1}} dy \right)^{1/p'}\end{aligned}$$

Morrey type inequality

Corollary.

Let X be a Banch space and $n < p < \infty, \gamma = 1 - n/p$. There exists a constant $C > 0$, depending on p and n such that for all $u \in C^1(\Omega, X)$

$$\|u\|_{C^{0,\gamma}(\Omega,X)} \leq C\|u\|_{W^{1,p}(\Omega,X)}$$