Jacobians of Sobolev homeomorphisms

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Problem

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- $\exists f$ homeomorphism, approximatively differentiable , $f(x) = x$ for $x \in \partial B(0, 1)$, but $J_f < 0$ has positive measure. (NOT $W^{1,1}$)

Results

Theorem

Let $\Omega \subset \mathbf{R}^n$ be an open set and $n \leq 3$. Suppose that $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$ is a homeomorphism. Then $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.

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Theorem

Let $\Omega \subset \mathbf{R}^n$ be an open set, $n \geq 2$. Suppose that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is a homeomorphism for some $p > [n/2]$. Then $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.

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Theorem

Let $\Omega \subset \mathbf{R}^n$ be an open set, $n \geq 2$. Suppose that $f : \Omega \to \mathbf{R}^n$ is a Sobolev homeomorphism with $\nabla f \in L_{p,1}$, where $p = [n/2]$. Then $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.

Tools from topology

Topological degree - $\deg(f,\Omega,y_0)=\sum_{\{x\in\Omega: f(x)=y_0\}}\operatorname{sgn}(J_f(x))$
$f:\Omega\to\mathbf{R}^n$ continuous is *sense-preserving* if
$\deg(f,\Omega',y_0)>0,\ \forall\Omega'\subset\subset\Omega$ and $\forall y_0\in f(\Omega')\setminus f(\partial\Omega')$.

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FACT 1: Each homeomorphism on a domain is either sense-preserving or sense-reversing.

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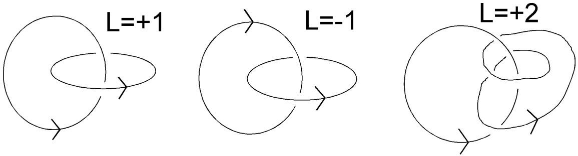
Linking number:



FACT 2: Linking number is a topological invariant.
If f is sense preserving, then it cannot map two curves with linking number $+1$ to curves with linking number -1.

Simple proof in dimension $n = 2$

Let f be a sense-preserving homeomorphism in $W^{1,1}(\Omega, \mathbf{R}^2)$. Let x_0 be a point, such that f is differentiable at x_0 and $J_f(x_0) < 0$.

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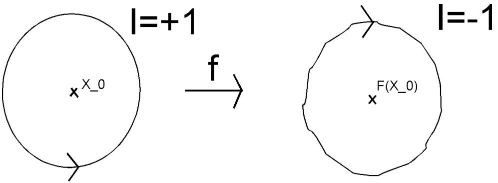

Index of a curve with respect to a point is a topological invariant - contradiction.

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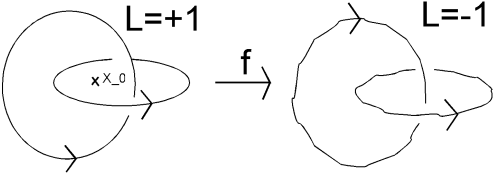

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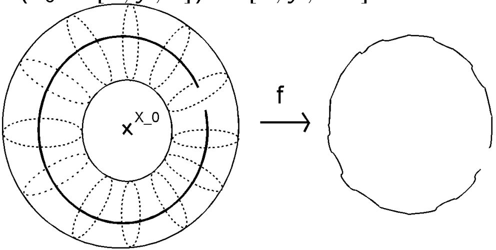

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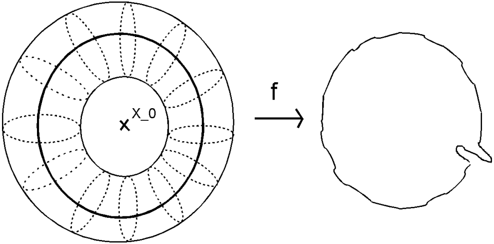

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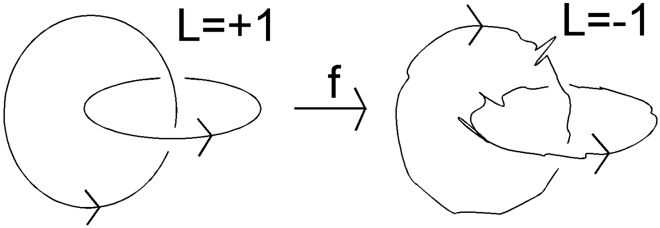

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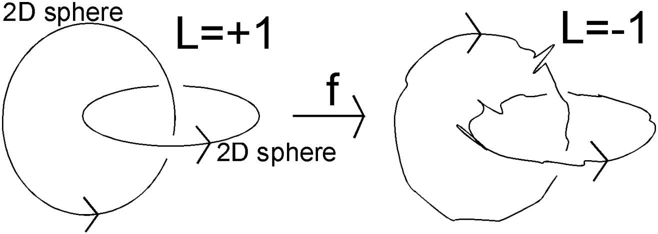

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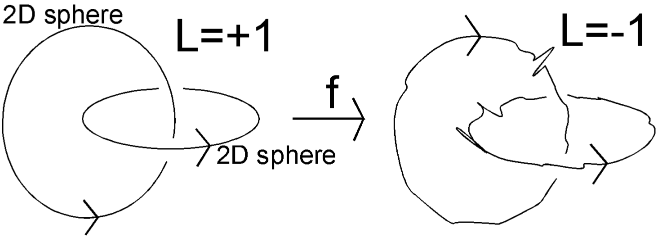

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$n = 4$ link one circle and one 2-dimensional sphere

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Question:

- $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ homeomorphism, $J_f \geq 0 \overset{?}{\Rightarrow} f$ is sense preserving
- $n \geq 2$, Does there exists a homeomorphism $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ such that $J_f = 0$ a.e.?

Homeomorphisms with $J_f \equiv 0$

f: $J_f = 0$ on a set of positive measure - Iteration does not work. Cannot have $J_f = 0 \Rightarrow |Df(x)| = 0$ a.e.

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