Analytic regularity and nonlinear approximation of a class of parametric semi-linear elliptic PDEs

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Semilinear elliptic PDEs

We are interested in the boundary value problem

$$-\nabla\big(a\nabla u\big)+G(u)=f \quad \text{in } D\,, \qquad u\big|_{\partial D}=0\,, \tag{1}$$

with solutions in $H_0^1(D)$.

Here $D\subset\mathbb{R}^n$ is a bounded Lipschitz domain, $f\in H^{-1}(D)$, and the composition operator $T_G: u\mapsto G(u)$ shall be a mapping from $H_0^1(D)$ into $H^{-1}(D)$. Of particular interest will be problems

$$-\nabla\big(a\nabla u\big)+u^m=f \quad \text{in } D\,, \qquad u\big|_{\partial D}=0\,,$$

where $m\in\mathbb{N}$ satisfies appropriate restrictions. For admissible parameters we will write $(n,m)\in\mathcal{M}$.

Motivation – PDEs with random input data

Let a probability space $(\Omega, \mathcal{A}, P)$ and a random field $a : \Omega \longrightarrow L_\infty(D)$ be given. Then $u(\omega)$ is defined as the solution of

$$-\nabla\big(a(\omega)\nabla u\big) + G(u) = f \quad \text{in } D\,, \qquad u\big|_{\partial D} = 0\,.$$

Under the assumption

$$0 < r \leq \operatorname{ess\,inf}_{x\in D} a(x,\omega) \leq a(x,\omega) \leq \operatorname{ess\,sup}_{x\in D} |a(x,\omega)| \leq R < \infty\,,$$

this solution is well-defined for every $\omega \in \Omega$, and the mapping $u : \Omega \longrightarrow H_0^1(D)$ is a well-defined random field.
The main task then consists in approximately computing this random field (or at least stochastical data such as expectations and other moments).

Reformulation as a parametric problem

Under appropriate assumptions the random field a admits an expansion

$$a(x,\omega) = \bar{a}(x) + \sum_{j\in\mathbb{N}} Y_j(\omega)\psi_j(x)\,,$$

e.g. the corresponding Karhunen-Loéve expansion. Here the Y_j are random variables with values in some bounded interval $[-R_j, R_j]$, and via re-scaling this may assumed to be the interval $[-1,1]$.

We now define

$$\widetilde{a}(x,y) = \bar{a}(x) + \sum_{j\in\mathbb{N}} y_j\psi_j(x)\,,$$

$$y \in U = [-1,1]^{\mathbb{N}} = \left\{ y = (y_j)_{j\in\mathbb{N}} : |y_j| \le 1 \right\}.$$

Then we re-obtain the random field a via $a(\omega) = \widetilde{a}(Y(\omega))$

Reformulation as a parametric problem II

Thus instead of solving the problem with random input data, we can consider the deterministic (!) parametric problem

$$-\nabla\big(\widetilde{a}(y)\nabla u\big) + G(u) = f \quad \text{in } D\,, \qquad u\big|_{\partial D} = 0\,,$$

for every fixed sequence of parameters $y \in U$. Denoting the parametric solution by $\widetilde{u}(y)$, we clearly re-obtain the random field u via $u(\omega) = \widetilde{u}(Y(\omega))$.

Hence instead of approximating $u(\omega)$, we can solve the initial task by approximating $\widetilde{u}(y)$.

Extension to complex parameters

We shall extend the parametric problem even further. For later arguments it will be convenient to include complex parameters, since the corresponding solution $u(z)$ then turns out to depend analytically on the parameters z.

Hence we put

$$\widetilde{a}(x,z) = \bar{a}(x) + \sum_{j\in\mathbb{N}} z_j\psi_j(x)\,,$$

on parameter domains

$$\mathcal{U}_\rho = \left\{z = (z_j)_{j\in\mathbb{N}} : |z_j| \leq \rho\right\},$$

where $\rho = (\rho_j)_{j\in\mathbb{N}}$ is a sequence of positive real numbers with $\rho_j \geq 1$. We shall write $\mathcal{U}$ for $\rho_j \equiv 1$.

Extension to complex parameters

We now define parameter domains

$$\mathcal{A}_\delta = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \delta \le \Re a(x,z) \le |a(x,z)| \le R(z) \text{ for a.e. } x \in D \right\},$$

where $\delta > 0$ is a fixed parameter.

The main assumption then reads as $\mathcal{U} \subset \mathcal{A}_\delta$ for some $\delta > 0$ (*Uniform ellipticity assumption* $UEA(\delta)$), i.e. we assume there exists some $\delta > 0$ such that for every $z \in \mathcal{U}$ we have the estimate

$$\begin{aligned} \delta \le \operatorname{ess\,inf}_{x \in D} |a(x,z)| &\le \Re a(x,z) \le |a(x,z)| \\ &\le \operatorname{ess\,sup}_{x \in D} |a(x,z)| < \infty . \end{aligned}$$

Existence of solutions

Theorem

Consider the problem (1) with $G(\zeta) = \zeta^m$ for $\zeta \in \mathbb{C}$ and $(n, m) \in \mathcal{M}$. Let a satisfy the uniform ellipticity assumption UEA(δ), and suppose

$$\|f|H^{-1}(D)\|^{m-1} < C_m \delta^m \,. \tag{2}$$

Then for every fixed parameter $z \in \mathcal{A}_\delta$, the problem (1) admits a uniquely determined solution $u(z) \in H_0^1(D)$, satisfying

$$\|u(z)|H_0^1(D)\| \leq \frac{2\|f|H^{-1}(D)\|}{\delta} \,. \tag{3}$$

Regularity of the parameter dependence

Theorem

Let $f \in H^{-1}(D)$ satisfy the scaling-condition (2). Then at any point $z \in \mathcal{A}_\delta$, the function $z \mapsto u(z)$ admits partial derivatives $\partial_{z_j} u(z) \in H_0^1(D)$ with respect to each complex variable z_j. For each $j \in \mathbb{N}$, $\partial_{z_j} u(z)$ is the unique solution in $H_0^1(D)$ of the linear problem:

$$-\nabla\big(a(z)\nabla w\big) + G'\big(u(z)\big)w = \partial_{z_j} f(z) + \nabla\big(\psi_j \nabla u(z)\big),$$

or equivalently

$$\int_D a(x,z)\nabla w(x)\cdot\nabla v(x)dx + \int_D G'\big(u(x,z)\big)w(x)v(x)dx = \int_D \partial_{z_j} f(x,z)v(x)dx - \int_D \psi_j(x)\nabla u(x,z)\cdot\nabla v(x)dx$$

for all $v \in H_0^1(D)$.

Based on this regularity result we can estimate the Taylor coefficients

$$t_\nu = \frac{1}{\nu!}\partial^\nu u(0) \in H_0^1(D)\,, \qquad \nu \in \mathcal{F}\,,$$

where $\mathcal{F} = \ell_1(\mathbb{N}) \cap \mathbb{N}_0^{\mathbb{N}}$ is the set of multiindices (i.e. sequences of natural numbers with finite support), and

$$\nu! = \prod_{j\in\mathbb{N}} \nu_j! = \prod_{j\in\operatorname{supp}\nu} \nu_j!\,.$$

Then we are interested in (absolute, uniform and/or unconditional) convergence of the Taylor expansion

$$\sum_{\nu\in\mathcal{F}} t_\nu z^\nu\,,$$

as well as estimates for the convergence rate of (optimal) partial sums (i.e. m-term approximation with respect to the sup-Norm on $\mathcal{U}$).

Theorem

Suppose f satisfies condition (2)*. Moreover, let* a *satisfy* $UEA(\delta)$*, and assume*

$$\big(\|\psi_j|L_\infty(D)\|\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N}) \quad \text{for some } p<1. \tag{4}$$

Then the Taylor coefficients t_ν *of the solution* u *of* (1) *satisfy* $\big(\|t_\nu|H_0^1(D)\|\big)_{\nu\in\mathcal{F}} \in \ell_p(\mathcal{F})$*. Moreover, it holds*

$$\sum_{\nu\in\mathcal{F}} t_\nu z^\nu = u(z)\,, \qquad z\in\mathcal{U}\,,$$

with absolute and uniform convergence. More precisely, if $(\Lambda_N)_{N\geq 1}$ *is a suitable sequence of subsets of* $\mathcal{F}$*, then the partial sums* $S_{\Lambda_N}u(z)=\sum_{\nu\in\Lambda_N} t_\nu z^\nu$ *satisfy*

$$\lim_{N\to\infty}\sup_{z\in\mathcal{U}} \|u(z)-S_{\Lambda_N}u(z)|H_0^1(D)\| = 0\,.$$

When talking about the parameter domain $U = [-1,1]^{\mathbb{N}}$, apart from Taylor series one is also interested in expansions with respect to (tensorized) Legendre and/or Chebyshev polynomials.
Here we only consider the Chebyshev polynomials $T_n(t) = \cos(n \arccos(t))$, $n \in \mathbb{N}$, and $T_0(t) = 1$. These satisfy

$$\| T_n | L_\infty([-1,1]) \| = 1 \, , \quad n \geq 0 \, ,$$

and

$$\int_{[-1,1]} | T_n(t) |^2 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2} \, , \quad n \in \mathbb{N} \, .$$

We consider the set U equipped with the Borel σ-algebra $\mathcal{B}(U)$ and the probability measure

$$d\eta = \bigotimes_{j \in \mathbb{N}} \frac{dt}{\pi \sqrt{1-t^2}} \, .$$

Then the tensorized system $\left(T_\nu\right)_{\nu\in\mathcal{F}}$ is orthogonal in $L_2(U, d\eta)$, where

$$T_\nu(z) = \prod_{j\geq 1} T_{\nu_j}(z_j)\,.$$

From this observation we immediately obtain the unique expansions

$$u(y) = \sum_{\nu\in\mathcal{F}} w_\nu\, T_\nu(y)\,, \qquad y \in U\,,$$

with convergence in $L_2(U, d\eta; H_0^1(D))$, where the $H_0^1(D)$-valued coefficients w_ν are given by

$$w_\nu = 2^{-|\operatorname{supp}(\nu)|} \int_U u(y)\, T_\nu(y) d\eta(y)\,.$$

Theorem

Let f fulfill condition (2), and suppose a satisfies UEA(δ). Moreover, assume

$$\left(\|\psi_j|L_\infty(D)\|\right)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N}) \quad \text{for some } p < 1. \tag{5}$$

Then the Chebyshev coefficients w_ν of the solution u of (1) satisfy $\left(\|w_\nu|H_0^1(D)\|\right)_{\nu\in\mathcal{F}} \in \ell_p(\mathcal{F})$. Furthermore, it holds

$$u(z) = \sum_{\nu\in\mathcal{F}} w_\nu T_\nu(z)\,, \qquad z \in \mathcal{U}\,,$$

with absolute and uniform convergence.

N-term approximation by partial sums

Theorem

Let a and f satisfy the assumptions as above. If we denote by $\Lambda_N^{(T)} \subset \mathcal{F}$ an index set corresponding to the N largest values of $\|t_\nu|H_0^1(D)\|$, then it holds

$$\sup_{y\in U}\left\|u(y) - S_{\Lambda_N^{(T)}}^{(T)} u(y)\middle| H_0^1(D)\right\| \leq \left\|\left(\|t_\nu|H_0^1(D)\|\right)_{\nu\in\mathcal{F}}\middle|\ell_p(\mathcal{F})\right\| N^{-s}\,,$$

where

$$s = \frac{1}{p} - 1\,.$$

This remains true for the Legendre Polynomials P_ν and the Chebyshev polynomials T_ν with the respective coefficients, partial sums and index sets for largest coefficients.

Moreover, if $\Lambda_N^{(C)} \subset \mathcal{F}$ *is a set corresponding to the* N *largest values of* $\|w_\nu | H_0^1(D)\|$*, then it holds*

$$\left\| u - S_{\Lambda_N^{(C)}}^{(C)} u \middle| L_2\big(U, d\eta; H_0^1(D)\big) \right\| \leq \left\| \big(\|w_\nu | H_0^1(D)\|\big)_{\nu\in\mathcal{F}} \middle| \ell_p(\mathcal{F}) \right\| N^{-s},$$

where now

$$s = \frac{1}{p} - \frac{1}{2},$$

and once more this remains true for Legendre polynomials P_ν *with respect to the uniform measure.*

Summary

The parametric solution $u(y) \in H_0^1(D)$ of the semilinear elliptic equation

$$-\nabla\big(a(y)\nabla u\big) + u^m = f \quad \text{in } D\,, \qquad u\big|_{\partial D} = 0\,,$$

can be approximated by absolutely and uniformly convergent partial sums

$$S_\Lambda^{(T)} u(y) = \sum_{\nu\in\Lambda} t_\nu y^\nu$$

with algebraic convergence rates independent of the number of parameters, and the coefficients itself are solutions of linear elliptic equations.