SOBOLEV SPACES OF

DIFFERENTIAL FORMS

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ABSTRACT

We study Sobolev type classes of differential forms $\Omega_{q,p}(M)$ on Riemannian manifolds that contains q-integrable forms ω with p-integrable weak exterior differential $d\omega$. Cohomology of the Banach complex $\{\Omega_{q,p}(M), d\}$ are invariants of Lipschitz structure. There exists a relation between these global cohomology classes (that we call $L_{q,p}$-cohomology) and a version of Sobolev inequality for $\Omega_{q,p}(M)$. This relation is correct for the classical Sobolev inequality also and corresponds to 1-dimensional $L_{q,p}$-cohomology.

We show some applications manifolds with pinched curvature and to quasi-linear equations.

$HISTORY \quad and \quad MOTIVATION$

The Sobolev spaces of differential forms $\Omega^k_{q,p}$ and L^*_p-cohomology were introduced at the beginning of 80's by G., Kuz'minov and Shvedov on Lipschitz manifolds as invardiants of Lipschitz structures on manifolds. M.Gromov and P.Pansu studied L^*_p-cohomology of infinte groups and Riemannian manifolds with pinched curvature (starting from 90's). At the same time S.Zucker studied L^*_p-cohomology of arithmetic manifolds.

The special case $p = q = \infty$, $M = R^n$ of the integration theory for the classes $\Omega^*_{\infty,\infty}(R^n)$ corresponds to "Geometrical Theory of Integration" by H.Whitney.

The case $p = q = 2$ coresponds to the well-known theory of L^*_2-cohomology.

SOBOLEV type INEQUALITY

In the paper of 2006 by G. and Troyanov a concept of $L^*_{q,p}$-cohomology was introduced. In the framework of a $L^*_{q,p}$-cohomology theory a version of Sobolev inequlities for differential forms (for $\Omega_{q,p}$-spaces) was introduced also. Roughly speacking existence of Sobolev inequalities for differential forms is equivalent to vanishing of corresponding $L^*_{q,p}$-cohomology.

A special class of (quasi)-conformally invariant $L^k_{\frac{m}{k},\frac{m}{k+1}}$-cohomology of m-dimensional Riemannian manifolds (conformal de Rham complex) was inrtroduced by G. and M.Troyanov in 2008.

Recent applications of $L_{q,p}$-cohomology to the Hodge theory, conformal structures on manifolds and quasi-linear equations were studied by Jammes, Kopylov, Li...

$Weak \quad exterior \quad differential$

We denote by $L^1_{loc}(M,\Lambda^k)$ the space of differential k-forms whose coefficients are locally integrable.

One says that a form $\theta \in L^1_{loc}(M,\Lambda^k)$ is the *weak exterior differential* of a form $\phi \in L^1_{loc}(M,\Lambda^{k-1})$ and one writes $d\phi = \theta$ if for each $\omega \in C^\infty_{comp}(M,\Lambda^{n-k})$, one has

$$\int_M \theta \wedge \omega = (-1)^k \int_M \phi \wedge d\omega \,.$$

$SOBOLEV \quad SPACES \quad of \quad forms$

Let $L^q(M, \Lambda^k)$ be a Banach space of k-dimensional q-integrable differential forms with the standard norm $\|\omega\|_{L^q(M,\Lambda^k)} := (\int_M |\omega|^p \, dvol)^{1/p}$ for $1 \leq p < \infty$ and $\|\omega\|_{L^\infty(M,\Lambda^k)} := \operatorname{ess\,sup}_{x \in M} |\omega(x)|$.

Denote by $\Omega^k_{q,p}(M)$ a Banach space of weakly differentiable differential forms $\omega \in L^q(M, \Lambda^k)$ with $d\omega \in L^p(M, \Lambda^{k+1})$.

A Banach complex $\left\{\Omega^k_{q,p}(M), d\right\}$ is Lipschitz invariant.

If $\dim M = m$, $q = \frac{m}{k}$, $p = \frac{m}{k+1}$ then a Banach graded algebra $\left\{\Omega^k_{\frac{m}{k},\frac{m}{k+1}}(M), d\right\}$ is (quasi)-conformal invariant.

Exact and closed forms

We set $Z^k_p(M) := L^p(M,\Lambda^k) \cap \ker d$ (= the set of weakly closed forms in $L^p(M,\Lambda^k)$) and

$$B^k_{q,p}(M) := d\left(L^q(M,\Lambda^{k-1})\right) \cap L^p(M,\Lambda^k)$$

(=the set of weakly exact forms in $\Omega^k_{q,p}(M)$

Observe that $B^k_{q,p}(M) \subset Z^k_p(M) \subset \Omega^k_{q,p}(M)$,

we thus have

$$B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M).$$

$SOBOLEV\ INEQUALITY\ for\ COMPACT$

$MANIFOLDS$

THEOREM. Let (M, g) be a smooth n-dimensional compact Riemannian manifold, $1 \leq k \leq n$ and $p, q \in (1, \infty)$. Then there exists such a constant C that for any smooth differential form θ of degree $k-1$ on M we have

$$\inf_{\zeta \in Z^{k-1}} \|\theta - \zeta\|_{L^q(M)} \leq C \|d\theta\|_{L^p(M)}$$

if and only if

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}.$$

It means that mainly interesting cases are complete unbounded manifolds or manifolds with singularities (domains with non Lipschitz boundaries)

$Non\ compact\ manifolds$

In the case of a non compact manifold, the inequality is still meaningful. Although the condition $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$ is still necessary in the non compact case, it is no longer sufficient and additional conditions must be imposed on the geometry of the manifold (M, g) for the Sobolev inequality to hold (here g is a Riemannian metric).

We give a necessary and sufficient condition based on an invariant called the *$L_{q,p}$–cohomology of Riemannian manifold (M, g).*

$\mathbf{L_{q,p}}$ – cohomology

The $L_{q,p}$-*cohomology* of (M, g) (where

$1 \leq p, q \leq \infty$) is defined to be the quotient

$$H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M)\,,$$

and the *reduced* $L_{q,p}$-*cohomology* of (M, g) is

$$\overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M)\,,$$

(where $\overline{B}^k_{q,p}(M)$ is the closure of $B^k_{q,p}(M)$).

When $p = q$, we simply speak of L_p-cohomology and write $H^k_p(M)$ and $\overline{H}^k_p(M)$.

$Compact\ manifolds,\ cohomological\ interpratation$

THEOREM. Let (M,g) be a smooth compact Riemannian manifold of dimension n and $p,q \in (1,\infty)$. There exists a constant C such that for all closed differential forms ω of degree k with coefficients in $L^p(M)$, there exists a differential form θ of degree $k-1$ such that $d\theta = \omega$ and

$$\|\theta\|_{L^q} \leq C\,\|\omega\|_{L^p}\,,$$

if and only if p,q satisfy the condition $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$ and $H^k_{\textit{DeRham}}(M,g) = 0$.

Non compact case

THEOREM. $H^k_{q,p}(M,g) = 0$ if and only if there exists a constant $C < \infty$ such that for any closed p-integrable differential form ω of degree k there exists a differential form θ of degree $k-1$ such that $d\theta = \omega$ and

$$\|\theta\|_{L^q} \leq C \, \|\omega\|_{L^p} \, .$$

TORSION

$$0 \to T^k_{q,p}(M,g) \to H^k_{q,p}(M,g) \to \overline{H}^k_{q,p}(M,g) \to 0$$

THEOREM. If $T^k_{q,p}(M) = 0$, then there exists a constant C' such that for any differential form $\theta \in \Omega^{k-1}_{q,p}(M)$ of degree $k-1$ there exists a closed form $\zeta \in Z^{k-1}_q(M)$ such that

$$\|\theta - \zeta\|_{L^q} \leq C' \, \|d\theta\|_{L^p} \, (1)$$

B) Conversely, if $1 < q < \infty$, and if there exists a constant C' such that for any form $\theta \in \Omega^{k-1}_{q,p}(M)$ of degree $k-1$ there exists $\zeta \in Z^{k-1}_q(M)$ such that (1) holds, then $T^k_{q,p}(M) = 0$.

Cohomology $H^k_{q,p}(M,g)$, $\overline{H}^k_{q,p}(M)$ are Lipschitz invariant.

Theorem. If $q = \frac{m}{k-1}$ and $p = \frac{m}{k}$, then

$H^k_{q,p}(M,g)$ and $\overline{H}^k_{q,p}(M,g)$ are (quasi)-conformal invariants.

The vanishing of torsion gives sufficient condition to solving the non linear equation

$$\delta(\|d\theta\|^{p-2}\, d\theta) = \alpha$$

where δ is the operator defined for $\omega \in L^1_{loc}(M, \Lambda^k)$ as

$$\delta\, \omega = (-1)^{nk+n+1} * d * \omega.$$

This operator is the formal adjoint to the exterior differential d in the sense that

$$\int_M \langle \omega, d\varphi \rangle \, d\mathrm{vol} = \int_M \langle \delta\omega, \varphi \rangle \, d\mathrm{vol}$$

for any $\varphi \in C_c^\infty(M, \Lambda^{k-1})$.

THEOREM. Assume $T^k_{q,p}(M) = 0$, $1 < q, p < \infty$ and $\alpha \in L^{q'}(M, \Lambda^k)$.

If $\int_M < \alpha, \varphi > dvol = 0$ for any $\varphi \in Z^k_q(M)$, then the equation $\delta(\|d\theta\|^{p-2}\, d\theta) = \alpha$ has a solution $\theta \in L^k_q(M)$ with $d\theta \in L^{k+1}_p(M)$.

(Here $< \alpha, \varphi > dvol := \alpha \wedge *\varphi$, where $*$ is the Hodge operator.)

Remark. (Sufficient conditions for $T^k_{q,p}(M) = 0$) If $H^k_{q,p}(M) = 0$ then $T^k_{q,p}(M) = 0$.

Examples. (1) Hyperbolic space $\mathbb{H}^n$.

(A) $H^k_{\frac{n}{k-1},\frac{n}{k}}(\mathbb{H}^n) = 0$ for any $1 < k < n$.

(B) If $q < \frac{n-1}{k-1} < p$, then $H^k_{q,p}(\mathbb{H}^n) \neq 0$.

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(2) Euclidian space R^n.

A) $H^k_{\frac{n}{k-1},\frac{n}{k}}(R^n) = 0$ for any $1 < k < n$.

B) For any $q \neq \frac{n}{k-1}, p \neq \frac{n}{k}$

THEOREM. Let (M,g) be an n-dimensional Cartan-Hadamard manifold with sectional curvature $K \leq -1$ and Ricci curvature $Ric \geq -(1+\epsilon)^2(n-1)$.

(A) Assume that

$$\frac{1+\epsilon}{p} < \frac{k}{n-1} \quad \text{and} \quad \frac{k-1}{n-1} + \epsilon < \frac{1+\epsilon}{q},$$

then $H^k_{q,p}(M) \neq 0$.

{Recall that a Cartan-Hadamard manifold is a complete simply-connected Riemannian manifold of non positive sectional curvature}

Example

SOL is the group of 3×3 real matrices of the form

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a solvable and unimodular three dimensional Lie group. It is diffeomorphic to R^3 (with coordinates x, y, z) and a left invariant Riemannian metric is $ds^2 = e^{-2z}dx^2 + e^{2z}dy^2 + dz^2$ its volume measure is given by $dxdydz$ and is bi-invariant.

Theorem $\mathbb{H}^3$ and SOL are not quasiconformally equivalent.

This is result is not easy to prove directly, because SOL and $\mathbb{H}^3$ are both conformally hyperbolic, and they both have exponential volume growth.

Non vanishing of conformal cohomology

Theorem Let (M, g) be a complete Riemannian manifold, and let $\varphi : M \to \mathbb{B}^n$ be a Lipschitz map such that

$$|\Lambda^k \varphi| \in L^p(M); |\Lambda^{n-k}\varphi| \in L^{q'}(M),$$

where $q' = q/(q-1)$, assume also that

$$\varphi^*\omega \in L^1(M); \int_M \varphi^*\omega \neq 0,$$

where $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is the standard volume form on $\mathbb{B}^n$. Then $H^k_{q,p}(M) \neq 0$.

For the conformal case $p = \frac{n}{k}$ and $q' = \frac{n}{n-(k-1)}$.

Example $M := \{R^n; r^\beta dx_1^2 + r^\alpha \sigma_{n-2} + dr^2\}$.

If

$$\alpha \left(\frac{n}{k} - 2 \right) - \beta \left(\frac{n}{k} - 1 \right) < -1$$

and

$$\alpha \left(\frac{n}{n-(k-1)} - 2 \right) + \beta < -1,$$

then $H^k_{conf}(M) \neq 0$.

Warped products (*example*)

By the warped product $[a,b) \times_f M$ we mean the product of $[a,b)$ and a Riemannian manifold (M,g) with the smooth warping function $f : M \to R_+$ endowed with Riemannian metric $dt^2 + f^2(t)g$.

THEOREM If

$$\int_a^b f^{(m-kp)}(t)dt = \infty$$

then $\bar{H}^k([a,b) \times_f M) = 0$.

Denote by

$$\chi(a,b,v) := sup_{\tau \in [a,b)} \left\{ I_p J_{p'} \right\}$$

where $I_p := \left| \int_\tau^b |v(t)|^p \, dt \right|^{1/p}$ and $J_{p'} := \left| \int_\tau^b |v_o(t)|^{-p'} \, dt \right|^{1}$

.

Theorem If $\chi(a, b, f^{\frac{m}{p}-k+1}) < \infty$ and

$$\int_a^b f^{(m-kp)}(t)dt = \infty$$

then $H^k([a, b) \times_f M) = 0$.

1. Gol'dshtein V. and Troyanov M., *Sobolev Inequality for Differential forms and $L_{q,p}$-cohomology,* Journal of Geom. Anal.(2006), **16,** No 4, 597-631.

2. Gol'dshtein V. and Troyanov M., *Conformal de Rham complex,* Journal of Geom. Anal.(2010), **20,** No 3, 651-669.