BOUNDEDNESS OF SOME OPERATORS FROM RIS TO MORREY SPACES

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1. Main notations

$(\Re, \mu)$ is measure space;

M is set of real-valued measurable functions over $(\Re, \mu)$;

$M_+ = \{ f \in M, \quad f \geq 0 \}, \quad M_0 = \{ f \in M, \quad |f| < \infty \}.$

Mapping $\rho: M_+ \to [0, \infty]$ is called a *function norm* (FN) if

(A1)
$$\begin{cases} \rho(f) = 0 \quad \Leftrightarrow \quad f = 0; \quad \rho(\alpha f) = \alpha \rho(f), \quad \alpha > 0; \\ \rho(f+g) \leq \rho(f) + \rho(g); \end{cases}$$

(A2) $\quad 0 \leq g \leq f \quad \Rightarrow \quad \rho(g) \leq \rho(f)$ - lattice property;

(A3) $0 \le f_k \uparrow f \quad \Rightarrow \quad \rho(f_k) \uparrow \rho(f)$ - Fatou property;

(A4) $\mu(\Omega) < \infty \quad \Rightarrow \quad \rho\left(\chi_{\Omega}\right) < \infty$ - simple functions;

(A5) $\mu(\Omega) < \infty \quad \Rightarrow \quad \int_{\Omega} f \, d\mu < c_{\Omega}\, \rho(f)$ - local integrability.

Let ρ be a FN. Then,

$$F = F(\rho) = \{ f \in \mathrm{M} : \quad \rho(|f|) < \infty \}$$

is called a *Banach function space* (BFS) with

$$\| f \|_F := \rho(|f|).$$

Let $(\Re, \mu) = (R^n, \mu_n)$. Introduce

$$\lambda_f(\tau) = \mu_n \{x \in R^n : \ |f(x)| > \tau\}, \ \tau \in R_+;$$

$$f^*(t) = \inf\{\tau > 0 : \ \lambda_f(\tau) \le t\}, \ t \in R_+;$$

$$f^\#(x) = f^*\left(V_n |x|^n\right), \quad x \in R^n.$$

BFS $E = E(R^n)$ is called a *rearrangement invariant space* (RIS) if

$$f^* \le g^*, \ g \in E \ \Rightarrow \ f \in E, \ \|f\|_E \le \|g\|_E.$$

Examples. BFS:

$$F = L_\theta\left(R_+ ;\ w\right),\quad 1 \le \theta \le \infty ;\quad 1/\theta + 1/\theta' = 1;$$

$$\| g \|_F = \left(\int_0^\infty | g\, w |^\theta \, d\tau \right)^{1/\theta} ,\quad 1 \le \theta < \infty ;$$

$$\| g \|_F = ess \sup_{R_+} | g\, w |,\quad \theta = \infty .$$

Note that,

$$\text{(A4)}\quad \Rightarrow \left(\int_0^t w^\theta \, d\tau \right)^{1/\theta} < \infty ,\quad \text{(A5)}\quad \Rightarrow \left(\int_0^t w^{-\theta'} \, d\tau \right)^{1/\theta'} < \infty ,$$

$\forall\, t \in R_+$. *These assumptions on* F *are too restricted for our application.*

RIS: $E = L_p\left(R^n\right), \quad 1 \le p \le \infty;$

$$\| f \|_E = \left(\int_{R^n} | f |^p \, dx \right)^{1/p} = \left(\int_0^\infty \left(f^*\right)^p dt \right)^{1/p},$$

RIS: *Lorentz spaces* $E = \Lambda_p(u); \quad E = \Gamma_p(u), \quad 0 < p \le \infty;$

$$\| f \|_\Lambda = \left(\int_0^\infty \left(f^*\right)^p u \; dt \right)^{1/p};$$

$$\| f \|_\Gamma = \left(\int_0^\infty \left(f^{**}\right)^p u \; dt \right)^{1/p}; \quad f^{**}(t) = t^{-1} \int_0^t f^* \, d\tau .$$

2. Morrey type spaces.

RIS $E = E(R^n)$, FS $F = F(R_+)$ with (A1), (A2).

Local Morrey:

$$LM_{EF} = \left\{ f \in E^{loc} : \ \| f \|_{LM_{EF}} = \left\| \| f \chi_{B_r} \|_E \right\|_F < \infty \right\};$$

Global Morrey: let $T_x f(y) = f(x+y)$, then

$$GM_{EF} = \left\{ f \in E^{loc} : \ \| f \|_{GM_{EF}} = \sup_{x \in R^n} \| T_x f \|_{LM_{EF}} < \infty \right\};$$

Here

$$B(x,r) = \{ y \in R^n : |y-x| < r \}, \ B_r = B(0,r).$$

Example. Classical Morrey spaces we obtain in the case (with $\theta=\infty$)

RIS: $E = L_p\left(R^n\right), \quad 1 \le p \le \infty;$

FS: $F = L_\theta\left(R_+;\ w\right), \quad 1 \le \theta \le \infty;\ w(r) = r^{-\lambda},\ \lambda \ge 0.$

General weights w: Burenkov, Guliev, Gogatishvili, Rzaev, Goldman, …

General Morrey: let $\mathfrak{I} = \{T\}$ be some family of operators

$$T:\ E^{loc}\left(R^n\right) \to E^{loc}\left(R^n\right)$$

with property $\mathfrak{I} \in (\mathfrak{R})$:

1) $I \in \mathfrak{I}$; 2) $(Tf)^{\#} = f^{\#},\ \forall T \in \mathfrak{I},\ f \in E^{loc}$.

$$M_{EF}(\mathfrak{I}) = \left\{ f \in E^{loc} : \ \| f \|_{M_{EF}(\mathfrak{I})} = \sup_{T \in \mathfrak{I}} \| T f \|_{LM_{EF}} < \infty \right\}.$$

Examples.

1) If $\mathfrak{I} = \{ I \}$ then $\mathfrak{I} \in (\mathfrak{R})$, and $M_{EF}(\mathfrak{I}) = LM_{EF}$.

2). If $\mathfrak{I} = \{ T_x : \ x \in R^n \}$ then $\mathfrak{I} \in (\mathfrak{R})$, *and* $M_{EF}(\mathfrak{I}) = GM_{EF}$.

3) More general, let $\mathfrak{I}_0 = \{ \tau : R^n \to R^n \}$ be *some family of measure-preserving bijections* such that $i \in \mathfrak{I}_0$, let

$\mathfrak{I} = \{ T_\tau : \ \tau \in \mathfrak{I}_0 \}$, where $(T_\tau f)(y) = f(\tau(y)), \quad y \in R^n$.

Then $\mathfrak{I} \in (\mathfrak{R})$.

Definition. *Rearrangement invariant analogue of Morrey type spaces*:

$$\tilde{M}_{EF} = \left\{ f \in E^{loc} : \ \| f \|_{\tilde{M}_{EF}} = \left\| \left\| f^{\#} \chi_{B_r} \right\|_E \right\|_F < \infty \right\};$$

Proposition 1. *Let* $E = E\left(R^n\right)$ *be an RIS,* *FS* $F = F\left(R_+\right)$ *satisfies* (A1), (A2), *and* $\mathfrak{I} \in (\mathfrak{R})$. *Then the following embeddings hold*

$$\tilde{M}_{EF} \subset M_{EF}\left(\mathfrak{I}\right) \subset LM_{EF} \ ,$$

with estimates of the norms

$$\| f \|_{LM_{EF}} \le \| f \|_{M_{EF}(\mathfrak{I})} \le \| f \|_{\tilde{M}_{EF}} , \ f \in \tilde{M}_{EF} \ .$$

Proposition 2. *Let* $E = E\left(R^n\right)$ *be an RIS, FS* $F = F\left(R_+\right)$ *satisfies* (A1), (A2), (A3) *and also*

(A4)' $\left\| \varphi_{E_2}\left(\left|B_\rho\right|\right)\chi_{[0,r)}(\rho)\right\|_F < \infty, \ \forall r \in R_+;$

(A5)' $\left\| \chi_{[r,\infty)}(\rho)\right\|_F < \infty, \ \forall r \in R_+ .$

Then the space $\widetilde{M}_{EF}$ *is an RIS; in particular* $\widetilde{M}_{EF} \neq \{0\}$ *.*

Remark. We shall see later that assumptions (A4)' and (A5)' do not restrict generality in our considerations.

3. Positive operators from RIS to Morrey.

Let $E_1 = E_1\left(R^n\right), E_2 = E_2\left(R^n\right)$ be RIS's ; FS $F = F\left(R_+\right)$ satisfies (A1), (A2). For operator, $A:\ E_1 \to E_2^{loc}$ introduce

$$\|A\|_0 = \|A\|_{E_1 \to LM_{E_2 F}} \ ; \ \|A\|_{(\mathfrak{I})} = \|A\|_{E_1 \to M_{E_2 F}(\mathfrak{I})} \ ;$$

$$\|A\|_{\#} = \|A\|_{E_1 \to \tilde{M}_{E_2 F}} \ .$$

Obviously,

$$\|A\|_0 \le \|A\|_{(\mathfrak{I})} \le \|A\|_{\#} .$$

Say that operator A *has property* $B_{\#}\left(c_0\right)$ *and write* $A \in B_{\#}\left(c_0\right)$ *if*

$$\forall\ f \in E_1 \quad \exists\ \tilde{f} : \left(\tilde{f}\right)^{\#} = f^{\#},\ \left(A f\right)^{\#} \le c_0\ A \tilde{f}\ .$$

Examples. 1. $A = I \in B_{\#}(1)$; here $\tilde{f} = f^{\#}$.

2. Hardy operator: $\left(P f\right)\left(t\right) = t^{-1} \int_{t_0}^{t} f\, d\tau\,, \quad t \in R^1$. Then,

$$P \in B_{\#}(1): \quad \left(P f\right)^{\#} \le P \tilde{f}\,, \text{ where } \tilde{f}\left(t\right) = f^{\#}\left(t - t_0\right), \quad t \in R^1.$$

3. Maximal operator:

$$\mathrm{M}f(x) = \sup_{r>0} |B_r|^{-1} \int_{B(x,r)} |f(y)|\,dy, \quad x \in R^n.$$

Then, $\mathrm{M} \in B_{\#}(c_0)$, $c_0 = c_0(n) \in [1, \infty)$; here $\tilde{f} = f^{\#}$;

$$c_0 = c_2 / c_1 : \qquad c_1 (\mathrm{M}f)^* \le f^{**} \le c_2 (\mathrm{M}f)^* \; .$$

Proposition 3. *For operator* $A \in B_{\#}(c_0)$ *the following estimate holds*

$$\|A\|_{\#} \le c_0 \|A\|_0 \; .$$

Corollary. *For* $A \in B_{\#}(c_0)$ *we have chain of inequalities*

$$\|A\|_0 \le \|A\|_{(\Im)} \le \|A\|_{\#} \le c_0 \|A\|_0 .$$

Remark. For maximal operator

$$\|\mathrm{M}\|_0 = \|\mathrm{M}\|_{E_1 \to GM_{E_2}F} \le \|\mathrm{M}\|_{\#} \le c_0 \|\mathrm{M}\|_0 .$$

Conclusion. Thus, *we will concentrate our attention on the estimates of*

$$\|A\|_{\#} = \|A\|_{E_1 \to \tilde{M}_{E_2}F} .$$

4. Estimates for $\|A\|_{\#}$.

In this Section we estimate $\|A\|_{\#}$ *under the following assumptions*:

$A:\ E_1 \to E_2^{loc}$*, where* E_1 *is a Banach space;* $E_2 = E_2\left(R^n\right)$ *is an RIS; FS* $F = F\left(R_+\right)$ *satisfies* (A1), (A2).

Introduce for $r \in R_+$

$$\Phi_A\left(r\right) = \|A\|_{E_1 \to E_2\left(B_r\right)} = \sup\left\{ \left\| \left(Af\right)^{\#} \chi_{B_r} \right\|_{E_2} \ :\ \|f\|_{E_1} \le 1 \right\}$$

Proposition 4. *Under above assumptions, for* $0<\rho<r<\infty$ *we have*

$$\Phi_A(\rho)\le\Phi_A(r)\le\Phi_A(\rho)\left[1+\frac{\varphi_{E_2}(|B_r|)}{\varphi_{E_2}(|B_\rho|)}\right],$$

where $\varphi_{E_2}(t)=\|\chi_\Omega\|_{E_2}$; $\mu_n(\Omega)=t\in R_+$ *is the fundamental function of RIS* $E_2=E_2(R^n)$. *In particular, if* $\Phi_A(\rho)<\infty$ *for some* $\rho\in R_+$, *then* $\Phi_A(r)<\infty$ *for all* $r\in R_+$.

Theorem . *Under above assumptions, the following estimate holds*

$$\sup_{r>0}\left[\Phi_A(r)\left\|\frac{\varphi_{E_2}(|B_\rho|)}{\varphi_{E_2}(|B_\rho|)+\varphi_{E_2}(|B_r|)}\right\|_F\right]\le\|A\|_\#\le\|\Phi_A\|_F$$

(norm in $\|\cdot\|_F$ is calculated with respect to the variable $\rho\in R_+$).

Example. Let here $F\left(R_+\right)=L_\infty(w)$. Then,

$$\|A\|_\# = \sup_{\rho\in R_+}\left[\Phi_A(\rho)\tilde{w}(\rho)\right],\quad \tilde{w}(\rho)= \operatorname{ess}\sup_{\xi\in[\rho,\infty)} w(\xi).$$

Remark.

$$\frac{\varphi_{E_2}\left(\left|B_\rho\right|\right)}{\varphi_{E_2}\left(\left|B_\rho\right|\right)+\varphi_{E_2}\left(\left|B_r\right|\right)} \cong \begin{cases} \varphi_{E_2}\left(\left|B_\rho\right|\right) \big/ \varphi_{E_2}\left(\left|B_r\right|\right), & \rho < r; \\ 1, & \rho \geq r. \end{cases}$$

We see from here that

$$\left\| A \right\|_{\#} < \infty \;\Rightarrow\; \left\| \varphi_{E_2}\left(\left|B_\rho\right|\right)\chi_{[0,r)}(\rho) \right\|_F < \infty, \;\forall\, r \in R_+; \qquad \text{(i)}$$

$$\left\| A \right\|_{\#} < \infty \;\Rightarrow\; \left\| \chi_{[r,\infty)}(\rho) \right\|_F < \infty, \;\forall\, r \in R_+. \qquad \text{(ii)}$$

Conclusion. Without loss of generality we can assume that (i) and (ii) are fulfilled. *Therefore, in our problem we can assume that the space* $\tilde{M}_{E_2 F}$ *is an RIS (see Proposition 2).*

Proposition 5. *Under above assumptions, let*

$$E_1 = E_1\left(R^n\right) \subset E_2^{loc}\left(R^n\right).$$

1). *If operator* $A:\ E_1 \to E_1$ *is bounded i.e.,*

$$\| A \|_{11} := \| A \|_{E_1 \to E_1} < \infty ,$$

then

$$\| A \|_{\#} \le \| A \|_{11} \| I \|_{\#} ,$$

where $I:\ E_1 \to \tilde{M}_{E_2 F}$ *is embedding operator.*

2). *If* I *is majored by* A*, i.e.,*

$$\exists d_0 \in R_+ : \quad \left\| f^{\#} \chi_{B_r} \right\|_{E_2} \le d_0 \left\| (Af)^{\#} \chi_{B_r} \right\|_{E_2}, \quad r \in R_+,$$

then

$$\| I \|_{\#} \le d_0 \| A \|_{\#} .$$

Example. *If maximal operator* $\mathrm{M} : E_1 \to E_1$ *is bounded then*

$$\| I \|_{\#} \le \| \mathrm{M} \|_{\#} \le \| \mathrm{M} \|_{11} \| I \|_{\#} .$$