Embeddings of Besov spaces with generalized smoothness into Lorenz spaces

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Outline

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- Ω – a Borel subset of $\mathbb{R}^n$.
- $\mathcal{M}_0(\Omega)$ – family of all complex-valued or extended real-valued (Lebesgue-)measurable functions defined and finite a.e. on Ω.
- $\mathcal{M}_0^+(\Omega)$ – the subset of $\mathcal{M}_0(\Omega)$ consisting of those functions which are non-negative a.e. on Ω.
- $\mathcal{M}_0(a,b)$ – $\mathcal{M}_0((a,b))$.
- $\mathcal{M}_0^+(a,b)$ – $\mathcal{M}_0^+((a,b))$.
- $\mathcal{M}_0^+(a,b;\uparrow)$ – $f \in \mathcal{M}_0^+(a,b)$ which are non-increasing on (a,b).
- $\mathcal{M}_0^+(a,b;\downarrow)$ – $f \in \mathcal{M}_0^+(a,b)$ which are non-decreasing on (a,b).

$$\|f\|_{r,\Omega} := \begin{cases} \left(\int_\Omega |f(t)|^r \, dt\right)^{1/r} & \text{if } 0 < r < \infty \\ \operatorname{ess\,sup}_{t\in\Omega} |f(t)| & \text{if } r = \infty \end{cases}$$

Slowly varying functions

Definition (slowly varying function)

Let (α, β) be one of the intervals $(0, \infty)$, $(0, 1)$ or $(1, \infty)$. A function $b \in \mathcal{M}_0^+(\alpha, \beta)$, $0 \not\equiv b \not\equiv \infty$, is said to be slowly varying on (α, β), $(b \in SV(\alpha, \beta))$, if, for each $\varepsilon > 0$, there are functions $g_\varepsilon \in \mathcal{M}_0^+(\alpha, \beta; \uparrow)$ and $g_{-\varepsilon} \in \mathcal{M}_0^+(\alpha, \beta; \downarrow)$ such that

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \quad \text{for all} \quad t \in (\alpha, \beta).$$

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Examples

Let $\alpha, \beta \in \mathbb{R}$:

- $b(t) = (1 + |\log t|)^\alpha (1 + \log(1 + |\log t|))^\beta$;
- $b(t) = \exp(|\log t|^\alpha)$, $0 < \alpha < 1$.

Lorentz space

For $f \in \mathcal{M}_0(\mathbb{R}^n)$, we define the *non-increasing rearrangement* f^* by

$$f^*(t) := \inf\{\lambda \geq 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|_n \leq t\}, \quad t \geq 0.$$

Definition (Lorentz space)

Given $q \in (0, \infty]$ and a non-negative measurable function ω on the interval $(0,1)$, the classical Lorentz space $\Lambda_q^{loc}(\omega)$ is defined to be the set of all measurable functions $f \in \mathbb{R}^n$ such that

$$\|f\|_{\Lambda_q^{loc}(\omega)} := \|\omega f^*\|_{q;(0,1)} < \infty.$$

In particular, putting $\omega(t) := t^{1/p-1/q}\, b(t)$, $t \in (0,1)$, where $b \in SV(0,1)$, we obtain the *Lorentz-Karamata space* $L_{p,q;b}^{loc}$.
Note that Lorentz-Karamata spaces involve as particular cases the generalized Lorentz-Zygmund spaces, the Lorentz spaces, the Zygmund classes and Lebesgue spaces

$\mathbf{B}^{0,b}_{pr}$

Given $f \in L_p$, $1 \le p < \infty$, the *first difference operator* Δ_h of step $h \in \mathbb{R}^n$ transforms f in $\Delta_h f$ defined by

$$(\Delta_h f)(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n,$$

whereas the *modulus of continuity* of f is given by

$$\omega_1(f,t)_p := \sup_{\substack{h \in \mathbb{R}^n \\ |h| \le t}} \|\Delta_h f\|_p, \quad t > 0.$$

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Definition (Besov spaces $\mathbf{B}_{p,r}^{0,b}$)

Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,1)$ be such that

$$\|t^{-1/r} b(t)\|_{r,(0,1)} = \infty. \tag{1}$$

The Besov space $B_{p,r}^{0,b} = B_{p,r}^{0,b}(\mathbb{R}^n)$ consists of those functions $f \in L_p$ for which the norm

$$\|f\|_{B_{p,r}^{0,b}} := \|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,1)} \tag{2}$$

is finite. (with the usual modifications if $r = \infty$).

Remark

(i) *When* (1) *does not hold* $B^{0,b}_{p,r} \equiv L_p$.
(ii) *An equivalent norm results on* $B^{0,b}_{p,r}(\mathbb{R}^n)$ *if the modulus of continuity* $\omega_1(f,\cdot)_p$ *in* (2) *is replaced by the k-th order modulus of continuity* $\omega_k(f,\cdot)_p$ *, where* $k \in \{2,3,4,\dots\}$.
(iii) *Let the function* $b \in SV(0,\infty)$ *satisfy*

$$\|t^{-1/r} b(t)\|_{r,(1,\infty)} < \infty. \tag{3}$$

Then the functional

$$\|f\|_p + \|t^{-1/r} b(t)\,\omega_1(f,t)_p\|_{r,(0,\infty)} \tag{4}$$

is an equivalent norm on $B^{0,b}_{p,r}(\mathbb{R}^n)$.
Note also that assumption (3) *is natural. Otherwise the space of all functions on* $\mathbb{R}^n$ *for which norm* (4) *is finite is trivial* (*that is, it consists only of the zero element*).

Embeddings $B^{0,b}_{p,r} \hookrightarrow \Lambda^{loc}_q(w)$

Theorem (Reduction Theorem)

Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Assume that* ω *is a non-negative measurable function on* $(0,1)$. *Then*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{5}$$

for all $f \in B^{0,b}_{p,r}$ *if and only if*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t (f^*(u))^p \, du \right)^{1/p} \right\|_{r,(0,1)}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$.

Given $p, r \in (0, \infty]$ and a non-negative measurable function ω on the interval $(0,1)$, the *local generalized gamma space* $G\Gamma^{loc}_{p,r}(\omega)$ is defined to be the set of all measurable functions $f \in \mathbb{R}^n$ such that

$$\|f\|_{G\Gamma^{loc}_{p,r}(\omega)} := \left\| \omega(t) \left(\int_0^t (f^*(u))^p \, du \right)^{1/p} \right\|_{r,(0,1)} < \infty$$

$$G\Gamma^{loc}_{1,r}(\omega) = \Gamma^{loc}_r(t\omega(t))$$

Theorem (Reduction Theorem)

Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Assume that* ω *is a non-negative measurable function on* $(0,1)$. *Then*

$$B^{0,b}_{p,r} \hookrightarrow \Lambda^{loc}_q(\omega)$$

if and only if

$$G\Gamma^{loc}_{p,r}(t^{-1/r} b(t^{1/n})) \hookrightarrow \Lambda^{loc}_q(\omega)$$

Theorem ($B^{0,b}_{p,r} \hookrightarrow L^{loc}_q(w)$)

Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Define*

$$b_r(t) := \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)}, \quad t \in (0,1). \tag{6}$$

Put $\rho = \infty$ *if* $p \le q$ *and define* ρ *by* $\frac{1}{\rho} = \frac{1}{q} - \frac{1}{p}$ *if* $q < p$. *Assume that* ω *is a non-negative measurable function on* $(0,1)$ *and put*

$$\Omega_q(t) := \|\omega(s)\|_{q,(0,t)}, \quad t \in (0,1].$$

(i) *Let* $1 \le r \le q \le \infty$. *Then inequality* (5) *holds for all* $f \in B^{0,b}_{p,r}$ *if and only if*

$$\Omega_q(1) + \|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)} \lesssim b_r(t) \quad \text{for all} \quad t \in (0,1).$$

(ii) *Let* $0 < q < r < \infty$. *Then inequality* (5) *holds for all* $f \in B^{0,b}_{p,r}$ *if and only if*

$$\Omega_q(1) + \int_0^1 \left(\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)}\right)^{\frac{qr}{r-q}} b_r(t)^{\frac{r^2}{q-r}} b(t^{\frac{1}{n}})^r \frac{dt}{t} < \infty.$$

(iii) *Let* $0 < q < r = \infty$. *Put*

$$b^{**}_\infty(t) := t^{-1}\int_0^t b_\infty(\tau)\, d\tau, \quad t \in (0,1).$$

Then inequality (5) *holds for all* $f \in B^{0,b}_{p,r}$ *if and only if*

$$\Omega_q(1) + \int_{(0,1)} \left(\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)}\right)^q d(b^{**}_\infty(t)^{-q}) < \infty.$$

Embeddings $B^{0,b}_{p,r} \hookrightarrow L^{loc}_{p,q;\tilde{b}}$

Theorem ($B^{0,b}_{p,r} \hookrightarrow L^{loc}_{p,q;\tilde{b}}$, $r \le q$)

Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$*. Let* $b \in SV(0,1)$ *satisfy* (1) *and let* b_r *be given by* (6)*. Define, for all* $t \in (0,1)$,

$$\tilde{b}(t) := \begin{cases} b_r(t)^{1-r/q+r/\max\{p,q\}}\, b(t^{1/n})^{r/q-r/\max\{p,q\}} & \text{if } r \neq \infty \\ b_\infty(t) & \text{if } r = \infty \end{cases}. \qquad (7)$$

Then the inequality

$$\|t^{1/p-1/q}\tilde{b}(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}}$$

holds for all $f \in B^{0,b}_{p,r}$ *if and only if* $q \ge r$.

Theorem ($B^{0,b}_{p,r} \hookrightarrow L^{loc}_{p,q;\tilde{b}}$, $r > q$)

Let $1 \le p < \infty$, $1 \le r \le q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Define* b_r *and* $\tilde{b}$ *by* (6) *and* (7).

(i) *Let* $\kappa \in \mathcal{M}^+_0(0,1;\downarrow)$. *Then the inequality*

$$\|t^{1/p-1/q}\tilde{b}(t)\kappa(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{8}$$

holds for all $f \in B^{0,b}_{p,r}$ *if and only if* κ *is bounded.*

(ii) *Let* $\kappa \in \mathcal{M}^+_0(0,1)$ *and* $q = \infty$. *Then inequality* (8) *holds for all* $f \in B^{0,b}_{p,r}$ *if and only if* $\|\kappa\|_{\infty,(0,1)} < \infty$.

Growth envelopes $\mathcal{E}_G(A)$

Definition (Growth envelope)

Let $(A, \|\cdot\|_A) \subset \mathcal{M}_0(\mathbb{R}^n)$ *be a quasi-normed space such that* $A \not\hookrightarrow L_\infty$. *A positive, non-increasing, continuous function* $\mathcal{E}_G(t)$ *defined on some interval* $(0, \varepsilon]$, $\varepsilon \in (0,1)$, *is called the* (local) growth envelope function *of the space* A *provided that*

$$\mathcal{E}_G(t) \approx \sup_{\|f\|_A \leq 1} f^*(t) \quad \text{for all} \quad t \in (0, \varepsilon].$$

Given a growth envelope function $\mathcal{E}_G(t)$ *of the space* A *(determined up to equivalence near zero) and a number* $u \in (0, \infty]$, *we call the pair* $(\mathcal{E}_G(t), u)$ *the* (local) growth envelope *of the space* A *when the inequality*

$$\left(\int_{(0,\varepsilon)} \left(\frac{f^*(t)}{\mathcal{E}_G(t)} \right)^q d\mu_H(t) \right)^{1/q} \lesssim \|f\|_A$$

(with the usual modification when $q = \infty$*) holds for all* $f \in A$ *if and only if the positive exponent* q *satisfies* $q \geq u$. *Here* μ_H *is the Borel measure associated with the non-decreasing function* $H(t) := -\ln \mathcal{E}_G(t)$, $t \in (0, \varepsilon)$. *The component* u *in the growth envelope pair is called the* fine index.

Growth envelopes in $B^{0,b}_{p,r}$

Theorem (Growth envelopes $\mathcal{E}_G(B^{0,b}_{p,r})$)

Let $1 \le p < \infty$, $1 \le r \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Define* b_r *by* (6). *Then*

$$\mathcal{E}_G(B^{0,b}_{p,r}) = (t^{-1/p}\, b_r(t)^{-1}, \max\{p,r\}).$$

Remark

(i) *Strictly speaking,* $t^{-\frac{1}{p}} b_r(t)^{-1}$ *might not have all the properties associated to a growth envelope function mentioned in* Definition *but, it is possible to show that there is always an equivalent function defined on* $(0,1)$*, namely,*

$$h(t) := \int_t^2 s^{-1/p-1}\, b_r(s)^{-1}\, ds,$$

which does.
(ii) *Since*

$$\|t^{1/p-1/q} b_r(t) f^*(t)\|_{q,(0,\varepsilon)} \lesssim \|f\|_{B^{0,b}_{p,r}} \quad \text{for all } f \in B^{0,b}_{p,r}$$

if and only if

$$q \geq \max\{p, r\}.$$

The embeddings of Besov spaces $B^{0,b}_{p,r}$ *into* $L^{loc}_{p,q;\tilde{b}}$ *cannot be described in terms of growth envelopes when* $1 \leq r \leq q < p < \infty$.

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The end...

Thank you for attention!