Function Spaces, Differential Operators and Nonlinear Analysis

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On Weyl numbers of Sobolev embeddings of weighted function spaces

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Introduction

Definition

Let $s \in \mathbb{R}$ *and* $1 \leq p, q \leq \infty$*. Then*

$$B^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \left\| f | B^s_{p,q} \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1} \varphi_j \mathcal{F} f \right\|_p^q \right)^{1/q} < \infty \right\}$$

is called Besov space. If $1 \leq p < \infty$ *then*

$$F^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \left\| f | F^s_{p,q} \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| \mathcal{F}^{-1} \varphi_j \mathcal{F} f(\cdot) \right|^q \right)^{1/q} \right\|_p < \infty \right\}$$

is called Triebel-Lizorkin space.
Here $\{\varphi_j\}$ *is a usually smooth dyadic partition of unity.*

- Let $p_1 \le p_2$, $s_1 \ge s_2$ and $A \in \{B, F\}$. It is well known that if $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) \ge 0$ and $q_1, q_2 \ge 1$ ($q_1 \le q_2$ if $\delta = 0$) then

$$A^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d),$$

  but the above embedding is never compact.

- We write $b \sim a$ if there exists constants $c, C > 0$ (independent of relevant parameters) such that

$$c\, a \le b \le C\, a\, .$$

- Let w_α be a positive smooth function such that

$$w_\alpha(x) \sim (1 + |x|^2)^{\alpha/2} \text{ for } \alpha > 0$$

  (polynomial weight),

  and let v_α be a positive smooth function such that

$$v_\alpha(x) \sim (e + \log|x|^2)^\alpha \quad \text{for sufficiently large} |x|, \alpha > 0$$

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Let $s, \in \mathbb{R}$ *and* $1 \leq p, q \leq \infty$*. Then for* $\omega \in \{w_\alpha, v_\alpha\}$ *define*

$$B^s_{p,q}(\mathbb{R}^d, \omega) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \omega f \in B^s_{p,q}(\mathbb{R}^d) \right\},$$

with the norm

$$\left\| f | B^s_{p,q}(\mathbb{R}^d, \omega) \right\| = \left\| \omega f | B^s_{p,q}(\mathbb{R}^d) \right\|.$$

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Theorem

Let $1 \le p_1, p_2 \le \infty, \quad 1 \le q_1, q_2 \le \infty, \quad -\infty < s_2 < s_1 < \infty$ *and* $\alpha > 0$*. Let* $\delta = s_1 - s_2 - d(\frac{1}{p_1} + \frac{1}{p_2})$ *and* $A \in \{B, F\}$*.*

(i) *(Haroske and Triebel in 2005)*
The embedding $A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ *is compact if and only if*

$$\min(\alpha, \delta) > d \max\left(\frac{1}{p_2} - \frac{1}{p_1}, 0\right). \tag{1}$$

(ii) *(Kühn, Leopold, Sickel, Skrzypczak in 2006)*
The embedding $A^{s_1}_{p_1,q_1}(\mathbb{R}^d, v_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ *is compact if and only if*

$$p_1 \le p_2 \quad \text{and} \quad \delta > 0. \tag{2}$$

Definition

Let X and Y be Banach spaces and $T \in L(X, Y)$.

(i) *For $k \in \mathbb{N}$, we define a kth aproximation number by*

$$a_k(T) := \inf\{\|T - A\| \; : \; A \in L(X, Y), \; \mathrm{rank}(A) < k\}, \tag{3}$$

where $\mathrm{rank}(A)$ *denotes the dimension of the range*

$$A(X) = \{A(x), x \in X\}$$

(introduced by Allakhverdiev in 1957).

(ii) *For $k \in \mathbb{N}$, we define a kth Weyl number by*

$$x_k(T) := \sup\{a_k(TS) : S \in L(\ell_2, X) \text{ with } \|S\| \le 1\}. \tag{4}$$

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The above numbers are examples of s-numbers, that were introduced by A. Pietsch.

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Properties

Let $T, S \in L(X, Y)$, X, Y *be Banach spaces and* $s_k \in \{a_k, x_k\}$, $k \in \mathbb{N}$:

- $\|T\| = s_1(T) \geq s_2(T) \geq s_3(T) \geq ... \geq 0$,
- $s_{k+n-1}(T + S) \leq s_n(T) + s_k(S)$ *for* $X \xrightarrow{T,S} Y$ *(additivity)*,
- $s_{k+n-1}(T \circ S) \leq s_n(T) \circ s_k(S)$ *for* $X \xrightarrow{T} Y \xrightarrow{S} Z$ *(multiplicativity)*,
- $x_k(T) \leq a_k(T)$,
- *if* $a_k(T) \to 0$ *then* T *is compact* ,
- *if* T *is compact then* $x_k(T) \to 0$.

Main results for polynomial weights

Theorem

Let $1 \le p_1, p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$, $-\infty < s_2 < s_1 < \infty$, $\alpha > 0$ $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2})$ *and* $A \in \{B, F\}$ *. Suppose that*

$$\min(\alpha,\delta) > d \max\left(\frac{1}{p_2} - \frac{1}{p_1}, 0\right) \quad \textit{and} \quad \alpha \neq \delta \quad \textit{then} \tag{5}$$

$$x_k\left(A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d)\right) \sim k^{-\beta},$$

$$\beta = \begin{cases} \frac{\min(\alpha,\delta)}{d} & \text{if} \quad 2 \le p_1 \le p_2 \le \infty, \\ \frac{\min(\alpha,\delta)}{d} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if} \quad 1 \le p_1, p_2 \le 2, \\ \frac{\min(\alpha,\delta)}{d} - \frac{1}{2} + \frac{1}{p_1}, & \text{if} \quad 1 \le p_1 < 2 < p_2 \le \infty, \\ \frac{p_1}{2}(\frac{\min(\alpha,\delta)}{d} + \frac{1}{p_1} - \frac{1}{p_2}), & \text{if} \quad \max(2,p_2) \le p_1 \text{ and } \min(\alpha,\delta) < \frac{d}{p_2}, \\ \frac{\min(\alpha,\delta)}{d} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if} \quad \max(2,p_2) \le p_1 \text{ and } \min(\alpha,\delta) > \frac{d}{p_2}. \end{cases}$$

Problem $\min(\alpha,\delta) = \frac{d}{p_2}$?

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Problem $\min(\alpha,\delta) = \frac{d}{p_2}$?

Main results for logarithmic weights

Theorem

Let $1 \le p_1 \le p_2 \le \infty,\ \ 1 \le q_1, q_2 \le \infty,\ \ -\infty < s_2 < s_1 < \infty,\ \ \alpha > 0,$ $A \in \{B, F\}$. *Suppose that*

$$\delta = s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0 \tag{6}$$

then

$$x_k\left(A^{s_1}_{p_1,q_1}(\mathbb{R}^d, v_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d)\right) \sim (1 + \log k)^{-\alpha} k^{-\beta},$$

$$\beta = \begin{cases} \frac{1}{p_1} - \frac{1}{p_2} & \text{if} \quad 1 \le p_1 \le p_2 \le 2, \\ \frac{1}{p_1} - \frac{1}{2} & \text{if} \quad 1 \le p_1 < 2 < p_2 \le \infty, \\ 0 & \text{if} \quad 2 \le p_1 \le p_2 \le \infty. \end{cases}$$

Methods of proof

Theorem (Haroske and Triebel)

Let $A \in \{B, F\}$. For every weight $\omega \in w_\alpha, v_\alpha$ there exists an orthonormal basis of compactly supported wavelet functions $\{\phi_{j,\ell}\}_{j,\ell} \cup \{\psi_{i,j,\ell}\}_{i,j,\ell}$, $j \in \mathbb{N}_0$, $\ell \in \mathbb{Z}$ and $i = 1, \dots 2^d - 1$, such that a distribution $f \in \mathcal{S}'$ belongs to $A^s_{p,q}(\mathbb{R}^d, \omega)$ if and only if

$$\left\| f \,|A^s_{p,q}(\mathbb{R}^d, \omega)\right\|^\clubsuit = \left(\sum_{\ell \in \mathbb{Z}^d} |\langle f, \phi_{0,\ell}\rangle\, \omega(\ell)|^p\right)^{1/p}$$

$$+ \sum_{i=1}^{2^d-1} \left\{ \sum_{j=0}^{\infty} 2^{j\left(s+d(\frac{1}{2}-\frac{1}{p})\right)q} \left(\sum_{\ell \in \mathbb{Z}^d} |\langle f, \psi_{i,j,\ell}\rangle\, \omega(2^{-j}\ell)|^p \right)^{q/p} \right\}^{1/q} < \infty.$$

Furthermore, $\left\| f \,|A^s_{p,q}(\mathbb{R}^d, \omega)\right\|^\clubsuit$ may be used as an equivalent quasi-norm on $A^s_{p,q}(\mathbb{R}^d, \omega)$.

It follows from this theorem that the Besov space $B^s_{p,q}(\mathbb{R}^d,\omega)$ and Triebel-Lizorkin $F^s_{p,q}(\mathbb{R}^d,\omega)$ are topologically isomorphic to the sequence space $\ell_q\Big(2^{j\sigma}\ell_p(\mathbb{Z}^d,\omega)\Big)$ with $\sigma = s + d(\frac{1}{2} - \frac{1}{p})$, where

$$\ell_q\Big(2^{j\sigma}\ell_p(\mathbb{Z}^d,\omega)\Big) := \Bigg\{\lambda = (\lambda_{j,k})_{j,k} : \quad \lambda_{j,k} \in \mathbb{C}\,,$$

$$\Big\|\lambda\,|\ell_q\Big(2^{j\sigma}\ell_p(\mathbb{Z}^d,\omega)\Big)\Big\| = \Bigg(\sum_{j=0}^{\infty} 2^{j\sigma q}\Bigg(\sum_{k\in\mathbb{Z}^d} |\lambda_{j,k}\,\omega(2^{-j}k)|^p\Bigg)^{q/p}\Bigg)^{1/q} < \infty\Bigg\}.$$

(i) Let $1 \leq p_1 < \max(2, p_2) \leq \infty$ and $1 \leq k \leq N/2$. Then

$$x_k(id : \ell_{p_1}^N \to \ell_{p_2}^N) \sim \begin{cases} k^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if} \quad 1 \leq p_1 \leq p_2 \leq 2, \\ 1, & \text{if} \quad 2 \leq p_1 \leq p_2 \leq \infty, \\ k^{\frac{1}{2} - \frac{1}{p_1}}, & \text{if} \quad 1 \leq p_1 < 2 < p_2 \leq \infty, \\ N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if} \quad 1 \leq p_2 < p_1 \leq 2. \end{cases}$$

(ii) Let $\max(2, p_2) \leq p_1$. Then

$$x_k(id : \ell_{p_1}^N \to \ell_{p_2}^N) \sim \begin{cases} N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if} \quad 1 \leq k \leq N^{\frac{2}{p_1}}, \\ N^{\frac{1}{p_2}} k^{-\frac{1}{2}}, & \text{if} \quad N^{\frac{2}{p_1}} \leq k \leq N, \end{cases}$$

(calculated by Pietsch and Lubitz).

- Let $w = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $w_{2k} \sim w_k$. For a real number $r > 0$ and any operators $T \in L(X, Y)$ we put

$$L^{(s)}_{r,w}(T) := \sup_{k\in\mathbb{N}} w_k^{1/r} s_k(T). \tag{7}$$

- The expression $L^{(s)}_{r,w}(T)$ is an example of a quasi-norm of an operator ideal. This means in particular that there exists a number $0 < \sigma \leq 1$ such that

$$L^{(s)}_{r,w}\Big(\sum_j T_j\Big)^{\sigma} \leq \sum_j L^{(s)}_{r,w}(T_j)^{\sigma}, \tag{8}$$

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Applications

Applications to spectral theory via Pietsch inequality :

- estimate asymptotic behaviour of eigenvalues of compact operators,
- estimate the size of the so-called negative spectrum of operators of a Schrödinger type.

The above applications can be found in a joint paper

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Thank you for your attention.