Integral Functionals on the Space of Regulated Functions with values in Banach Algebras

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Abstract—**In this paper we deal with the notion of regulated functions with values in a Banach algebra $\mathcal{A}$, we prove some results and present examples in two situations: using the set of continuous complex valued functions on a $[a,b]$, denoted by $C([a,b],\mathbb{C})$, and using a Banach algebra $\mathcal{S}_2(\mathbb{R})$ of $M_2(\mathbb{R})$, the space of real square matrix. Some particular results are presented, we consider then the Dushnik integral for these functions and we construct a correspondent linear integral functional on the Banach algebra of all regulated functions $G([a,b],\mathcal{A})$. Finally we present some examples in the spaces $G([a,b],C([a,b],\mathbb{C}))$ and $G([a,b],\mathcal{S}_2(\mathbb{R}))$.**

Keywords: Regulated Function, Banach Algebras, Integral Functionals

1 Introduction

Sometimes to describe physical events we need a model that has, besides the basic operations of linear spaces and the notion of size of their elements, an internal multiplication completely compatible with the normed linear space structure. These spaces are known as Banach algebras, subject that was treated by J. von Neumann, I. M. Gelfand and M. A. Naimark, among others, in the years 1930-60. For details see ([2]). Our interest here is to study the set of all well-behaved funtions $f : [a,b] \to \mathcal{A}$, i. e., the set of all functions that have the finite one-sided limits $f(t-)$, for every $t \in]a,b]$, and $f(t+)$, for every $t \in [a,b[$, (known as regulated functions) when $\mathcal{A}$ is a Banach algebra. The definition of regulated function first appeared in Dieudonne's book [1]. The space of regulated functions was approached by several authors, see for example, ([5] [6] [9]). The classical notation for this set of functions is $G([a,b],\mathcal{A})$ and it is a Banach space with the uniform convergence norm. In Section 2 we present the notions of regulated functions, Dushnik integral and Banach algebras, and we present proofs of some results to guarantee that $G([a,b],\mathcal{A})$ inherits the structure of A, in other words, it is also a Banach algebra.

The space of continuous functions $C(Z,\mathbb{C})$, if Z is a compact Hausdorff space, is considered in section 3. This is the most representative example of commutative Banach algebra because its universality (every Banach space is isomorphic to a subspace of some $C(Z)$). Here we consider $Z = [a,b]$. The other example envolves the Banach algebra $\mathcal{S}_2(\mathbb{R})$ of all square matrices $M = (a_{ij})_{1\le i,j\le 2}$, such that $a_{11} = a_{22}$ and $a_{21} = 0$, with the norm $\|M\| = |a_{11}| + |a_{12}|$. This is will be considered in Section 4.

Finally, in Subsections 3.2 and 4.2, the notions and results are then applied in the special cases when $G([a,b],C([a,b],\mathbb{C}))$ and $G([a,b],\mathcal{S}_2(\mathbb{R}))$, with the presentation of some examples of integral functionals.

2 Regulated Functions Banach Algebra Valued

Roughly speaking, algebras are simultaneously normed linear spaces and rings. They are structures with an addition, a scalar multiplication, an internal multiplication and a norm, all completly compatible. Formally we have that

Definition 1 *A Banach algebra A over the complex number field $\mathbb{C}$ is a structure $(A,+,\cdot,\times,\|\cdot\|)$ such that*

a. $(A,+,\cdot,\|\cdot\|)$ *is a Banach complex linear space;*

b. $(A,+,\cdot,\times)$ *is a algebra;*

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c. it is satisfied the submultiplicity condition

$$\|x \times y\| \le \|x\| \, \|y\|, \;\; \forall x, y \in A.$$

While the condition [c.] ensures that the internal multiplication is a continuous operation, the condition [b.] says that it is associative, that is, for all $x, y, z \in A$

$$x \times (y \times z) = (x \times y) \times z \ ,$$

and that are obeyed all the compatibility conditions:

. $(x + y) \times z = x \times z + y \times z$ and $x \times (y + z) = x \times y + x \times z$

. $\lambda \cdot (x \times y) = (\lambda \cdot x) \times y = x \times (\lambda \cdot y)$

If A contains an element e such that $e \times x = x \times e = x$, for every $x \in A$, and $\|e\| = 1$, we say that A is a Banach algebra with unit. When $x \times y = y \times x$, for every $x, y \in A$, we say that A is a commutative Banach algebra.

Definition 2 *We say that $f : [a, b] \to X$ is a regulated function if there exist both one-sided limits*

$$f(t+) = \lim_{\eta \downarrow t} f(\eta) \ , \ \textit{for every } t \in [a, b[\ ,$$

and

$$f(t-) = \lim_{\eta \uparrow t} f(\eta) \ , \ \textit{for every } t \in]a, b] \ .$$

We denote by $G([a, b], X)$ the Banach space of all X-valued regulated functions on $[a, b]$, with the uniform convergence norm $\|f\|_\infty = \sup\{\|f(t)\|_X, \ t \in [a, b]\}$. We begin recalling that in [4] it was proved that the multiplication on X induces an internal multiplication in $G([a, b], X)$.

Lemma 1 *Let f and g be two regulated functions on $[a, b]$ with values in a Banach algebra A. Then the pointwise multiplication $[f \times g](t) = f(t) \times_A g(t)$, $t \in [a, b]$ is a regulated function on $[a, b]$.*

As a consequence we have that the structure of Banach algebra is transferred to the space of regulated functions.

Theorem 1 *Suppose that A is a complex Banach algebra with multiplication $x \times y$. Then $G([a, b], A)$ with multiplication $[f \times g](t) = f(t) \times g(t)$ is a complex Banach algebra too.*

We note that if A is commutative Banach algebra, then $G([a, b], A)$ is also commutative.

Let A, B be two Banach algebras with multiplications $\times_A$ and $\times_B$ respectively. We consider the complex linear space

$$\mathcal{L}(A, B) = \{T : A \to B : \ T \text{ is a bounded linear operator}\},$$

with the usual norm

$$\|T\| = \sup\left\{\|T(x)\|_B \ : \ x \in A, \ \|x\|_A \le 1\right\} \ , \tag{1}$$

and, if $T, S \in \mathcal{L}(A, B)$, we define the multiplication as

$$[T \cdot S](x) = T(x) \times_B S(x) \tag{2}$$

Lemma 2 *If A, B are two Banach algebras, then $\mathcal{L}(A,B)$ with the usual norm and the multiplication (2) is a Banach algebra. If B is a commutative algebra, so $\mathcal{L}(A,B)$ is.*

Proof: See [4] for details.

Besides, when $B = A$ the composition of operators $T \circ S$ is the natural multiplication on $\mathcal{L}(A)$. In this case,

$$\|T \circ S\| = \sup \{\|[T \circ S](x)\| \ : \ x \in A, \ \|x\| \leq 1\} \leq \|T\|\|S\|$$

and we recall that if $dimA > 1$, then $\mathcal{L}(A)$ is non-commutative algebra.

While bounded linear functionals is an important notion in the study of Banach spaces, for Banach algebras the important notion is of multiplicative linear functionals (that are always continuous).

Definition 3 *A complex linear functional ϕ on a Banach algebra A is said multiplicative if $\phi(x\ y) = \phi(x)\ \phi(y)$, for every $x, y \in A$, and $\phi(e_{_A}) = 1$.*

We present now the notion of integral (in sense of Dushnik) that we use to decribe a perfomance criterion on the Banach algebra of regulated function. This kind of Stieltjes integral, finest than the Riemann-Stieltjes integral, is a convenient choice because, when the integrand function belongs to $G([a,b],A)$ and the integrator function is of bounded semivariation, the integral exists. The original definition of the Riemann integral has been modified in several different extensions. T. J. Stieltjes generalized the Riemann integral defining an integration of a continuous integrand with respect a bounded variation integrator, instead of the variable of integration. B. Dushnik in turn considered an integrand modification that consists in restricting integrand values only to the open segments of corresponding partitions of the interval $[a,b]$. This is a special case of the weighted refinement integral.

Let A, B be two Banach algebras with multiplications $\times_{_A}$ and $\times_{_B}$ respectively and suppose that $\alpha \in SV([a,b], \mathcal{L}(A,B))$, the Banach space of all bounded semivariation functions $\alpha : [a,b] \to \mathcal{L}(A,B)$, and $f \in G([a,b],A)$. Then there exists the Dushnik integral (see [6] for details)

$$F_\alpha(f) = \int_a^b \cdot\, d\, \alpha(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i^{\cdot}), \tag{3}$$

where $\xi_i^{\cdot} \in]t_{i-1}, t_i[$. Here the limit is taken over the set of all partitions of the interval $[a,b]$, denoted by $\mathcal{D}_{_{[a,b]}}$, and (3) means that, for all neighborhood V of a vector x, there is a partition $d_V \in \mathcal{D}_{_{[a,b]}}$ such that $d \leq d_V$ implies $x_d \in V$.

In [6] Hönig shows that if $dim\ B < \infty$, then $SV([a,b], \mathcal{L}(A,B)) = BV([a,b], \mathcal{L}(A,B))$, the Banach space of bounded variation functions.

Note that $F_\alpha : G([a,b],A) \to B$ is a linear map between the Banach algebras $G([a,b],A)$ and B. Moreover it has sense to write and to ask about

$$F_\alpha(f) \times_{_B} F_\alpha(g) = \int_a^b \cdot\, d\, \alpha(t) \cdot f(t) \times_{_B} \int_a^b \cdot\, d\, \alpha(t) \cdot g(t) \in B.$$

$$F_\alpha(f \times_{_G} g) = \int_a^b \cdot\, d\, \alpha(t) \cdot [f \times_{_G} g]\,(t)$$

In general

$$F_\alpha(f) \times_{_B} F_\alpha(g) \neq F_\alpha(f \times_{_G} g)$$

that is, F_α is not multiplicative, or F_α is not a homomorphism of Banach algebras. However, if we choose the integrator function conveniently (3) becames a homomorphism. Let $T \in \mathcal{L}(A,B)$ be a fix linear multiplicative operator. We denote by $\mathcal{K}^T([a,b],\mathcal{L}(A,B))$ the set of all functions $\alpha_c^T : [a,b] \to \mathcal{L}(A,B)$, $a \le c \le b$, defined as $\alpha_c^T = \mathcal{X}_{]c,b]}T$,

$$[\mathcal{X}_{]c,b]}T](t) = \begin{cases} 0 \,, t \in [a,c] \,, \\ \\ T \,, t \in]c,b] \,. \end{cases}$$

where $\mathcal{X}$ is the characteristic function. Of course we have $\mathcal{K}^T([a,b],\mathcal{L}(A,B)) \subset SV([a,b],\mathcal{L}(A,B))$, and if $d : t_0 < t_1 < ... < t_n$ is a partition of $[a,b]$ with $t_{k-1} = c$, for some $1 \le k \le n$. Then

$$\sum_{i=1}^{|d|}[\alpha_c^T(t_i) - \alpha_c^T(t_{i-1})]\ f(\xi_i^\cdot) = [\underbrace{\alpha_c^T(t_k)}_{T} - \underbrace{\alpha_c^T(t_{k-1})}_{0}] \cdot f(\xi_k^\cdot)$$

and

$$\int_a^b \cdot\, d\, \alpha_c^T(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|}[\alpha_c^T(t_i) - \alpha_c^T(t_{i-1})] \cdot f(\xi_i^\cdot) = T \cdot f(c^+)$$

So for $f, g \in G([a,b],A)$,

$$F_{\alpha_c^T}(f \times_G g) = \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot [f \times_G g]\,(t) = T \cdot [f \times_G g]\,(c^+) = T \cdot \left[f(c^+) \times_A g(c^+)\right] =$$

$$= T \cdot \left[f(c^+)\right] \times_B T \cdot \left[g(c^+)\right] = \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot f(t) \times_B \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot g(t) = F_{\alpha_c^T}(f) \times_B F_{\alpha_c^T}(g)$$

that is, $F_{\alpha_c^T} : G([a,b],A) \to B$ is a homomorphism of Banach algebras. Of course, $F_{\alpha_c^T}(e_G) = 1$.

In particular, if $T \in A^* = \mathcal{L}(A,\mathbb{C})$ is a fix linear multiplicative functional. We denote by $\mathcal{K}^T([a,b],A^*)$ the set of all functions $\alpha_c^T : [a,b] \to A^*$, $a \le c \le b$, defined as $\alpha_c^T = \mathcal{X}_{]c,b]}T$,

$$[\mathcal{X}_{]c,b]}T](t) = \begin{cases} 0 \,, t \in [a,c] \,, \\ \\ T \,, t \in]c,b] \,. \end{cases}$$

where $\mathcal{X}$ is the characteristic function. In this case we have $\mathcal{K}^T([a,b],A^*) \subset BV([a,b],A^*)$ (recall that is true because $dim\ B < \infty$), and then

$$\int_a^b \cdot\, d\, \alpha_c^T(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|}[\alpha_c^T(t_i) - \alpha_c^T(t_{i-1})] \cdot f(\xi_i^\cdot) = T \cdot f(c^+) \in \mathbb{C}$$

So for $f, g \in G([a,b],A)$,

$$F_{\alpha_c}(f \times_G g) = \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot [f \times_G g]\,(t) = T \cdot [f \times_G g]\,(c^+) = T \cdot \left[f(c^+) \times_A g(c^+)\right] =$$

$$= T \cdot \left[f(c^+)\right] \times_B T \cdot \left[g(c^+)\right] = \int_a^b \cdot\, d\, \alpha_c(t) \cdot f(t) \times_B \int_a^b \cdot\, d\, \alpha_c(t) \cdot g(t) = F_{\alpha_c^T}(f) \times_B F_{\alpha_c^T}(g)$$

that is, $F_{\alpha_c^T} : G([a,b],A) \to \mathbb{C}$ is a multiplicative linear functional on the Banach algebra $G([a,b],A)$. Another special case is when $B = G([a,b],A)$ and $F_\alpha(f) \in G([a,b],A)$,

$$[F_\alpha(f)](s) = \int_a^s \cdot\, d\, \alpha(t) \cdot f(t) \in A \ , \quad s \in [a,b] \ ,$$

or still $B = \mathbb{R}$ and $F_\alpha : G([a,b],C([a,b],\mathbb{R})) \in \mathbb{R}$,

$$F_\alpha(f) = \int_a^b \cdot\, d\, \alpha(t) \cdot f(t) \in \mathbb{R} \ . \tag{4}$$

Our first example of this situation was choosing by A the non-commutative algebra of quaternions, because the application in modelling of 3-D rotations (see [4]). Now the special cases $G([a,b],C(Z,\mathbb{C}))$, where $C(Z,\mathbb{C})$ is the set of all continuous complex valued functions and $G([a,b],\mathcal{S}_2(\mathbb{R}))$, where $\mathcal{S}_2(\mathbb{R})$ is the set of all triangular matrices $M = (a_{ij})_{1\le i,j\le 2}$ with $a_{11} = a_{22}$ and $a_{21} = 0$, will be considered in the next sections, where we present two known examples of Banach algebras and use its to show some regulated functions with values in this sets.

3 Continuous Functions

Let Z be a compact Hausdorff space. We denote by $C(Z,\mathbb{C})$ the set of all continuous complex valued functions $x : Z \to \mathbb{C}$, with the norm $\|x\|_\infty = \sup\{|x(s)| : s \in Z\}$. If X is a Banach space, we denote the unit ball of X by $B_X(0;1) = \{x \in X : \|x\| \le 1\}$. Recall that Alaoglu's Theorem ensure that $B_{X^*}(0;1)$ is a compact set of the dual space X^*. A theorem due to S. Banach shows, using the Alaoglu's Theorem, that $C(Z,\mathbb{C})$ is a universal Banach space in the sense that every Banach space is isomorphic to a subspace of some $C(Z,\mathbb{C})$. That is

Theorem 2 *(Banach) Every Banach space $\mathcal{X}$ is isometrically isomorphic to a closed subspace of $C(Z,\mathbb{C})$ for some compact Hausdorff space Z.*

and its proof can be find in [2]. This justifies our interest in those vector functions with values in this most representative example of Banach space. Observe that $C(Z,\mathbb{C})$ is an commutative algebra with the pointwise multiplication and the supremum norm satisfies the condition (c) of definition 1, i. e, $\forall x, y \in C(Z,\mathbb{C})$,

$$\|x \times y\|_\infty\| \le x\|_\infty \|y\|_\infty.$$

So $C(Z,\mathbb{C})$ is an commutative Banach algebra, with unit $e_z(z) = 1$, $\forall z \in Z$. As a consequence of Theorem 1 follows that $G([a,b],C(Z,\mathbb{C}))$ is a Banach algebra.

Here we start the application of the notions and results of the previous sections in the special case when $Z = [a,b]$ and $\mathcal{A} = C([a,b],\mathbb{C})$. We consider regulated functions as in section (2), i.e, we look to the Banach algebra $G([a,b],C([a,b],\mathbb{C}))$. More specifically we are interested in examples of linear integral functionals of kind (4). Theorem 5.1 of [6] shows that all linear functionals on $G_-([a,b],C([a,b],\mathbb{C}))$, the space of the left continuous regulated functions, can be represented by a bounded variation function $\alpha \in BV_0([a,b],\mathcal{L}(C([a,b],\mathbb{C}),\mathbb{R}))$. So we will take $p \in G([a,b],C([a,b],\mathbb{C}))$, $\alpha \in BV([a,b],\mathcal{L}(C([a,b],\mathbb{C}),\mathbb{R}))$ and

$$F_\alpha(p) = \int_a^b \cdot\, d\, \alpha(t) \cdot p(t) \in \mathbb{R} \ .$$

3.1 Functions in $G([a,b],C([a,b],\mathbb{C}))$

Let $p:[a,b]\to C([a,b],\mathbb{C})$ be a function. Then, for $t\in[a,b]$,

$$p(t):[a,b]\to\mathbb{C}$$

is a continuous function. We observe that

$$\|p\|_\infty=\sup\{\|p(t)\| \;:\; t\in[a,b]\}$$

where $\|p(t)\|=\sup\{|p(t)(s)| \;:\; s\in[a,b]\}$. Suppose that $p,q:[a,b]\to C([a,b],\mathbb{C})$ are two regulated functions. Then by Lemma 1 we have that $p\times q$ is a regulated function, that is, $p\times q\in G([a,b],C([a,b],\mathbb{C}))$. Moreover $[p(t)](s)=[p_1](t)(s)+i\,[p_2](t)(s)$ and we denote the multiplicative inverse of $p\in G([a,b],C([a,b],\mathbb{C}))$ by $1/p$,

$$[(\frac{1}{p})(t)](s)=[\frac{1}{p(t)}](s)\ ,[p(t)](s)\neq 0\ ,\ t,s\in[a,b]\ ,$$

and define

$$[\bar{p}(t)](s)=\overline{[p(t)](s)}$$

Example 1 Let $p:t\in[a,b]\mapsto w\in C([c,d],\mathbb{C})$ be a constant function defined as $p(t)=w$, where $w:[c,d]\to\mathbb{C}$

a. is the constant function $w(s)=z_0\in\mathbb{C}$.

b. is the function $w(s)=x(s)+i\ y(s)\in\mathbb{C}$, $s\in[c,d]$. In this case,

$$\|p\|_\infty=\sup_{a\le t\le b}\|p(t)\|_\infty=\sup_{a\le t\le b}\ \{\sup_{c\le s\le d}|[p(t)](s)|\}=$$

$$=\sup_{a\le t\le b}\ \{\sup_{c\le s\le d}|w(s)|\}=\sup_{a\le t\le b}\ \{\sup_{c\le s\le d}\sqrt{x^2(s)+y^2(s)}\}$$

c. $x(s)=y(s)=1$ and $p(t)=sin(t/2\pi)\ w$, $t\in[0,1]$.

d. $w:[0,2\pi]\to\mathbb{C}$ is the function $w(s)=\cos(s)+i\ \sin(s)\in\mathbb{C}$ and $p:t\in[0,1]\mapsto t\ w\in C([0,2\pi],\mathbb{C})$.

$$\|p\|_\infty=\sup_{0\le t\le 1}\ \{\sup_{0\le s\le 2\pi}t\ \sqrt{x^2(s)+y^2(s)}\}=1$$

3.2 Integral Functionals on $G([a,b],C([a,b],\mathbb{C}))$

Some functionals are classics in the theory of regulated functions with values in Banach algebras. For example, if we consider $c\in[a,b[$ we can define $p(c+):s\in[a,b]\mapsto[p(c+)](s)\in\mathbb{C}$, where $p(c+)=\lim_{\eta\downarrow c}p(\eta)$, $p(c+)(s)=p_1(c+)(s)+i\ p_2(c+)(s)$ and

$$F:G([a,b],C([a,b],\mathbb{C}))\longrightarrow\mathbb{R}\ ,$$

where $F(p)=Re[p(c+)(b)]$. Another cases are $F(p)=\|p(b-)-p(a+)\|$, $F(p)=p(a+)$ or $F(p)=\|p\|_{C([a,b],\mathbb{C})}$.

Example 2 For functionals of the form (4) we take $\alpha:[a,b]\to\mathcal{L}(C([a,b],\mathbb{C}),\mathbb{R})$

$$\alpha(t) = \begin{cases} F \ , \ t \in [a,c] \\ \\ 0 \ , \ t \in]c,b]. \end{cases}$$

where

$$F : z \in C([a,b],\mathbb{C}) \longmapsto Re\left[\int_a^b z(s)\ ds\right] \in \mathbb{R}$$

We recall (3) and we choose the partition $d : a = t_0 < t_1 < .. < t_n = b$ such that $t_k = c$. Then we have $\beta_c(t_i) - \beta_c(t_{i-1}) = 0$, if $i \neq k+1$, and $\beta_c(t_{k+1}) - \beta_c(t_k) = 0 - F = -F$.

So, if $z : [a,b] \to C([a,b],\mathbb{C})$ is a regulated function,

$$\sum_{i=1}^{n} [\beta_c(t_i) - \beta_c(t_{i-1})] \cdot z(\xi_i^{\cdot}) = [\beta_c(t_{k+1}) - \beta_c(t_k)] \cdot z(\xi_i^{\cdot}) = -F \cdot z(\xi_k^{\cdot})$$

where $\xi_k^{\cdot} \in]t_k, t_{k+1}[$. Now we have that

$$\mathcal{F}_{\beta_c}(z) = \int_a^{b} \cdot\ d\ \beta_c(t) \cdot z(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\beta_c(t_i) - \beta_c(t_{i-1})] \cdot z(\xi_i^{\cdot}) = -F \cdot z(c^+) = -Re\left[\int_a^b z(c^+)(s)\ ds\right]$$

4 Matrices

Let $\mathcal{S}_2(\mathbb{R})$ be the set of all matrices of the form

$$M = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix} \ ,$$

where $p, q \in \mathbb{R}$, with the usual multiplication of matrices and norm $\|M\| = |p| + |q|$. $\mathcal{S}_2(\mathbb{R})$ is the subspace

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right]$$

The submultiplicity condition is satisfied: If $M, N \in \mathcal{S}_2(\mathbb{R})$,

$$\|M \times N\| = \left\| \begin{pmatrix} p_1 & q_1 \\ 0 & p_1 \end{pmatrix} \ \begin{pmatrix} p_2 & q_2 \\ 0 & p_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} p_1\ p_2 & p_1\ q_2 + p_2\ q_1 \\ 0 & p_1\ p_2 \end{pmatrix} \right\| = |p_1\ p_2| + |p_1\ q_2 + p_2\ q_1|$$

$$\leq |p_1\ p_2| + |p_1\ q_2| + |p_2\ q_1| \leq |p_1\ p_2| + |p_1\ q_2| + |p_2\ q_1| + |q_1\ q_2|$$

$$= |p_1|\,(|p_2| + |q_2|) + |q_1|\,(|p_2| + |q_2|) = (|p_1| + |q_1|)\ \ (|p_2| + |q_2|) = \|M\|\ \|N\|$$

$\mathcal{S}_2(\mathbb{R})$ is a 2-dimensional commutative Banach algebra.

4.1 Functions in $G([a,b],\mathcal{S}_2(\mathbb{R}))$

Let $M:[a,b]\to\mathcal{S}_2(\mathbb{R})$ be a function. Then

$$M(t)=\begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix},$$

where p,q are $\mathbb{R}$-valued functions, $t\in[a,b]$.

Lemma 3 *If $p,q\to\mathbb{R}$ are two regulated functions, then $M:[a,b]\to\mathcal{S}_2(\mathbb{R})$ defined as*

$$M(t)=\begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix},$$

is a regulated function, that is, $M\in G([a,b],\mathcal{S}_2(\mathbb{R}))$.

Proof: Since p,q are regulated functions, for every $\epsilon>0$ there are partitions $R: a=r_0<r_1<..<r_m=b$ and $S: a=s_0<s_1<..<s_n=b$ of $[a,b]$ such that

$$\|p(\xi)-p(\eta)\|<\frac{\epsilon}{2}\ ,\quad \text{for all } r_{j-1}<\xi<\eta<r_j\ , j=1,2,..,m$$

and

$$\|q(\xi)-q(\eta)\|<\frac{\epsilon}{2}\ ,\quad \text{for all } s_{k-1}<\xi<\eta<s_k\ , k=1,2,..,n$$

Take the partition $T=R\cup S$, $T: a=t_0<t_1<..<t_l=b$, and $\xi,\ \eta\in]t_{i-1},t_i[$, $i=1,2,..,l$. Then

$$\|M(\xi)-M(\eta)\|=\|\begin{pmatrix} p(\xi) & q(\xi) \\ 0 & p(\xi) \end{pmatrix}-\begin{pmatrix} p(\eta) & q(\eta) \\ 0 & p(\eta) \end{pmatrix}\|=\|\begin{pmatrix} p(\xi)-p(\eta) & q(\xi)-q(\eta) \\ 0 & p(\xi)-p(\eta) \end{pmatrix}\|$$

$$=|p(\xi)-p(\eta)|+|q(\xi)-q(\eta)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

Note: In this case

$$\|M\|_\infty=\sup\{\|M(t)\|\ :\ t\in[a,b]\}=\sup\{|p(t)|+|q(t)|\ :\ t\in[a,b]\}$$

A typical example of this kind of function

$$M_1(t)=\begin{pmatrix} t^2 sin(1/t) & cos(t) \\ 0 & t^2 sin(1/t) \end{pmatrix},$$

If $M\in G([a,b],\mathcal{S}_2(\mathbb{R}))$,

$$M(t)=\begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix},$$

M is invertible iff $p(t)\neq 0$, for all $t\in[a,b]$.

4.1.1 Integral functionals on $G([a,b],\mathcal{S}_2(\mathbb{R}))$

The following result can be used to find integral linear functionals (see [6]):

If $\mathcal{F} : G^{-}([a,b],\mathcal{S}_2(\mathbb{R})) \to \mathbb{R}$ is a bounded linear functional, there is a bounded variation function $\beta : [a,b] \to \mathcal{L}(\mathcal{S}_2(\mathbb{R})),\mathbb{R})$ such that

$$\mathcal{F}(M) = \int_a^b \cdot\, d\, \beta(s) \cdot M(s)$$

Example 3 Consider the linear functional on $\mathcal{S}_2(\mathbb{R})$ defined by $Tr : M \in \mathcal{S}_2(\mathbb{R}) \mapsto Tr(M) \in \mathbb{R}$, where $Tr(M)$ is the trace of M. Then $Tr(M) = 2\ p$, if

$$M = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix} ,$$

If $M \in G([a,b],\mathcal{S}_2(\mathbb{R}))$ we have

$$M(t) = \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} .$$

Take now, for $c \in [a,b]$, the function $\beta_c : [a,b] \to \mathcal{L}(\mathcal{S}_2(\mathbb{R})),\mathbb{R})$ as

$$\beta_c(t) \cdot M = \begin{cases} Tr\ M,\ t \in [a,c]\ , \\ \\ 0\ ,\ t \in]c,b]\ . \end{cases}$$

We choose the partition $d : a = t_0 < t_1 < .. < t_n = b$ such that $t_{k-1} = c$. Then we have $\beta_c(t_i) - \beta_c(t_{i-1}) = 0$, if $i \neq k$. So, if $M : [a,b] \to \mathcal{S}_2(\mathbb{R})$ is a regulated function,

$$\sum_{i=1}^{n} [\beta_c(t_i) - \beta_c(t_{i-1})] \cdot M(\xi_i^{\cdot}) = [\underbrace{\beta_c(t_k)}_{0} - \underbrace{\beta_c(c)}_{Tr}] \cdot M(\xi_i^{\cdot}) = -Tr \cdot M(\xi_k^{\cdot})$$

where $\xi_k^{\cdot} \in]t_{k-1}, t_k[$. Now we have that

$$\mathcal{F}_{\beta_c}(M) = \int_a^b \cdot\, d\, \beta_c(t) \cdot M(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\beta_c(t_i) - \beta_c(t_{i-1})] \cdot M(\xi_i^{\cdot}) = -Tr \cdot M(c^+) = -2\ p(c^+)$$

If

$$M_1(t) = \begin{cases} \begin{pmatrix} t^2 sin(1/t) & cos(t) \\ 0 & t^2 sin(1/t) \end{pmatrix},\ t \neq 0\ , \\ \\ 0,\ t = 0. \end{cases}$$

$$\mathcal{F}_{\beta_c}(M_1) = \int_a^b \cdot\, d\, \beta_c(t) \cdot M_1(t) = -2\ \lim_{\tau \downarrow c} \tau^2 sin(1/\tau)$$

Let $A \in M_2(\mathbb{R})$ be a fixed matrix and consider the linear operator on $\mathcal{S}_2(\mathbb{R})$ defined as $T_A(M) = A\ M$. That is, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } M = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix} \ , \ \text{ then } T_A(M) = \begin{pmatrix} a\,p & a\,q + b\,p \\ c\,p & c\,q + d\,p \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } M = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix} \ , \ \text{ then } Tr\ T_A(M) = Tr\ \begin{pmatrix} a\,p & a\,q + b\,p \\ c\,p & c\,q + d\,p \end{pmatrix} = (a+d)\ p + c\ q$$

Example 4 We know that all linear functional $T : \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is defined by

$$T : \begin{pmatrix} p & q \\ 0 & p \end{pmatrix} \in \mathcal{S}_2(\mathbb{R}) \longmapsto r\ p + s\ q \in \mathbb{R}.$$

for some $r, s \in \mathbb{R}$. We denote by $T_{r,s}$ such functional, to a fixed couple of real numbers r, s, and consider $\gamma_c : [a, b] \to \mathcal{L}(\mathcal{S}_2(\mathbb{R}), \mathbb{R})$,

$$\gamma_c(t) \cdot M = \begin{cases} 0\ ,\ t \in [a, c[\ , \\ \\ T_{r,s}\ M\ ,\ t \in [c, b]\ . \end{cases} = \begin{cases} 0\ ,\ t \in [a, c[\ , \\ \\ r\ p + s\ q\ ,\ t \in [c, b]\ . \end{cases}$$

As in example 3, we choose the partition $d : a = t_0 < t_1 < .. < t_n = b$ such that $t_k = c$. We have then $\gamma_c(t_i) - \gamma_c(t_{i-1}) = 0$, if $i \neq k$. Then, if $M : [a, b] \to \mathcal{S}_2(\mathbb{R})$ is a regulated function,

$$\sum_{i=1}^{n} [\gamma_c(t_i) - \gamma_c(t_{i-1})] \cdot M(\xi_i^\cdot) = [\underbrace{\gamma_c(t_k)}_{T_{r,s}} - \underbrace{\gamma_c(t_{k-1})}_{0}] \cdot M(\xi_i^\cdot) = T_{r,s} \cdot M(\xi_k^\cdot)$$

where $\xi_k^\cdot \in]t_{k-1}, t_k[$. Now we have that

$$\mathcal{F}_{\gamma_c}(M) = \int_a^b \cdot\ d\ \gamma_c(t) \cdot M(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\gamma_c(t_i) - \gamma_c(t_{i-1})] \cdot M(\xi_i^\cdot) = T_{r,s} \cdot M(c^-) = r\ p(c^-) + s\ q(c^-)$$

5 Conclusions and Future Work

This paper is part of an effort to find examples of functionals on the Banach algebra of regulated functions with values on Banach algebras. In the future we hope there will be a conection of this subject with optimization of this kind of functionals when they are convex, or Lipschitz, for example.

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