Resolvent Estimates and Dissipativity of Mixed Order Systems

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Resolvent estimates for scalar elliptic BVPs

$$A = \sum_{|\alpha| \le 2m} a_\alpha D_x^\alpha, \qquad x \in \mathbb{R}^n_+$$

$$B = (B_1, \ldots, B_m)^\top, \quad \operatorname{ord} B_j = r_j, \quad r_1 \le \ldots \le r_m \le 2m - 1$$

Theorem

Let A_B be parameter-elliptic on a sector $\mathcal{L}$.

- $\mathcal{L} \supset \mathbb{C}_+ \quad \Longrightarrow \quad$ *A_B generates analytic semigroup on*

$$X^s := \left\{ u \in H^s_p(\mathbb{R}^n_+) \colon B_j A^k u = 0, \quad r_j + 2mk < s - 1/p \right\}.$$

- $s > r_1 + 1/p \quad \Longrightarrow \quad$ *A_B generates no C_0 semigroup on $H^s_p(\mathbb{R}^n_+)$, $B^s_{pp}(\mathbb{R}^n_+)$ (Nesensohn 2009)*
- $A = \triangle, \quad B_1 = 1 \quad \Longrightarrow$

$$\left\| (A_B - \lambda)^{-1} \right\|_{\mathcal{L}(W^1_p(\mathbb{R}^n_+))} \ge \frac{C}{|\lambda|^{(p+1)/(2p)}}$$

Minimal growth of the resolvent

$A = (a_{jk}(x, D))_{j,k=1,\dots,N}$, $\qquad \operatorname{ord} a_{jk} \le m$

$B = (B_1, \dots, B_{mN/2})^\top$, $\quad \operatorname{ord} B_j = r_j \le m - 1$,

parameter-elliptic in $\mathcal{L} \supset \mathbb{C}_+$

Theorem (Denk, D. 2010)

Let $\gamma \in (-\infty, 1]$, $s \ge 0$, $1 < p < \infty$, and fix $f \in \mathcal{K}_p^s(\Omega) \in \{H_p^s(\Omega), B_{pq}^s(\Omega)\}$.
Assume that λ_0 and C_0 exist such that $u := (A_B - \lambda)^{-1} f$ satisfies

$$|\lambda|^\gamma \|u\|_{\mathcal{K}_p^s(\Omega)} \le C_0 \|f\|_{\mathcal{K}_p^s(\Omega)}$$

for all $\lambda \in \mathcal{L}$ with $|\lambda| \ge \lambda_0$.
Then $\gamma_0 B_j f \equiv 0$ for all j with

$$\gamma > \frac{m + r_j + 1/p - s}{m}.$$

Sketch of proof

$$
\begin{aligned}
|\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \left\|\gamma_0 B_j f\right\|_{L^p(\partial\Omega)} &= |\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \left\|\gamma_0 B_j A u\right\|_{L^p(\partial\Omega)} \\
&\le C\left(\left\|B_j A u\right\|_{\mathcal{K}_p^{s-r_j}(\Omega)} + |\lambda|^{\frac{s-r_j}{m}} \left\|B_j A u\right\|_{\mathcal{K}_p^{0}(\Omega)}\right) \\
&\le C\left(\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \, \|u\|_{\mathcal{K}_p^{s}(\Omega)}\right) \\
&\le C|\lambda|^{1-\gamma} \|f\|_{\mathcal{K}_p^{s}(\Omega)} \, .
\end{aligned}
$$

Conclusion for $A_B = \triangle_D$:

$$
\left\|(A_B - \lambda)^{-1}\right\|_{\mathcal{L}(W_p^1(\mathbb{R}_+^n))} \le \frac{C_\varepsilon}{|\lambda|^{(p+1)/(2p)+\varepsilon}} \qquad \text{impossible.}
$$

Elliptic systems of uniform order

$A = (a_{jk}(x,D))_{j,k=1,\dots,N}, \qquad \operatorname{ord} a_{jk} \le m$

$B = (B_1, \dots, B_{mN/2})^\top, \quad \operatorname{ord} B_j = r_j \le m-1,$
parameter-elliptic in $\mathcal{L} \supset \mathbb{C}_+$

Theorem (Denk, D. 2010)

Let $s > 0$, $1 < p < \infty$, and let Y^s be any closed subspace of $\mathcal{K}_p^s(\Omega)$, and

$$D(A_B) := \left\{u \in Y^s \cap W_p^m(\Omega) \colon Au \in Y^s, \quad B_j u = 0\right\}.$$

Then the following are equivalent:

- A_B *generates analytic semigroup on* Y^s
- $D(A_B) \subset Y^s$ *dense,* $(A_B - \lambda)^{-1} \in \mathcal{L}(Y^s)$ *for all* $\lambda \in \mathcal{L}$ *with* $|\lambda| \gg 1$, *and* $B_j A^k u = 0$ *for all* $u \in Y^s$ *and all* $r_j + mk < s - 1/p$.

Parameter-elliptic Douglis–Nirenberg systems

$A = A(x, D) = (a_{jk}(x, D))_{j,k=1,\dots,N}$ with $\operatorname{ord} a_{jk} \leq s_j + m_k$

Theorem (Denk, Saal, Seiler 2009)

If A is parameter-elliptic in $\mathcal{L} \subset \mathbb{C}$, then A admits a bounded $\mathcal{H}_\infty$ calculus on $W^s_p(\mathbb{R}^n)$.

Boundary value problems

$$\begin{cases} (A-\lambda)u = f & \text{in } \Omega, \qquad \lambda \in \mathcal{L} \\ Bu = g & \text{on } \partial\Omega \end{cases}$$

the orders on the diagonal of A shall be equal:
$s_1 + m_1 = s_2 + m_2 = \ldots = s_N + m_N =: m$.

$B = (b_{jk})_{j,k}$ with $j = 1, \ldots, mN/2 =: q, \qquad k = 1, \ldots, N$

$b_{jk} = \gamma_0 \sum_\beta b_{jk\beta}(x) D_x^\beta$

$\operatorname{ord} b_{jk} \leq r_j + m_k$ with $r_j \in \mathbb{Z}$

Well-posedness and estimates

Theorem (Agranovich, Grubb, Solonnikov, Volevich,...)

If (A, B) *is parameter-elliptic,* $\min m_j = 0$, $\max r_j \leq m - 1$, *then:*

$\exists\, \lambda_0 \quad \forall\, \lambda \in \mathcal{L}, \quad |\lambda| \geq \lambda_0$:
$\forall\, f \in \prod_j W_p^{m_j}(\Omega), \quad \forall\, g \in \prod_j W_p^{m-r_j-1/p}(\partial\Omega)$
$\exists!$ *solution* $u \in \prod_j W_p^{m+m_j}(\Omega)$,

and we have the estimate

$$\sum_{j=1}^{N} \left(\|u_j\|_{W_p^{m+m_j}} + |\lambda|^{1+m_j/m} \|u_j\|_{L^p} \right)$$

$$\leq C \sum_{j=1}^{N} \left(\|f_j\|_{W_p^{m_j}} + |\lambda|^{m_j/m} \|f_j\|_{L^p} \right)$$

$$+ C \sum_{j=1}^{mN/2} \left(\|g_j\|_{W_p^{m-r_j-1/p}} + |\lambda|^{\frac{m-r_j-1/p}{m}} \|g_j\|_{L^p} \right)$$

Resolvent estimates for DN systems: necessary conditions

$A = (a_{jk}(x, D))_{j,k=1,\dots,N}$, $\quad \operatorname{ord} a_{jk} \le s_j + m_k$, $\quad s_j + m_j = m$,

$B = (B_1, \dots, B_{mN/2})^\top = (b_{jk})_{j,k}$, $\quad \operatorname{ord} b_{jk} \le r_j + m_k$, $\quad r_j \in \mathbb{Z}$

$\min_j m_j = 0$, $\quad r_j \le m - 1$

Theorem (Denk, D. 2010)

Suppose that (A, B) is parameter-elliptic in Ω with sector $\mathcal{L}$. Assume that $f \in W_p^{m_1} \times \dots \times W_p^{m_N}$ is such that a constant C_0 exists with

$$\sum_{j=1}^{N} \left(\|u_j\|_{W_p^{m+m_j}(\Omega)} + |\lambda| \, \|u_j\|_{W_p^{m_j}(\Omega)} \right) \le C_0 \sum_{j=1}^{N} \|f_j\|_{W_p^{m_j}(\Omega)}$$

for all solutions u to $(A - \lambda)u = f$, $Bu = 0$.
Then $B_j f \equiv 0$ for all j with $r_j \le -1$.

Resolvent estimates for DN systems: sufficient conditions (example)

$$m = 4, \qquad N = 4$$

$$\vec{m} = \begin{pmatrix} 4 & 3 & 3 & 0 \end{pmatrix}$$

$$\operatorname{ord} A = \begin{pmatrix} 4 & 3 & 3 & 0 \\ 5 & 4 & 4 & 1 \\ 5 & 4 & 4 & 1 \\ 8 & 7 & 7 & 4 \end{pmatrix}, \qquad \vec{s} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}$$

Special structure of boundary condition

$$B = \begin{pmatrix} \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & \bullet \end{pmatrix} = \text{diag}(B_1, B_2, B_3)$$

assume:
$\exists$ A_1, scalar PDO of order $m = 4$,
$\exists$ A_2, 2×2 PDO of order $m = 4$,
such that (A_1, B_1) and (A_2, B_2) are parameter-elliptic

Main Result

Choose the ground space

$$X = \left\{ f \in W_p^4 \times W_p^3 \times W_p^3 \times L^p : b_j f = 0 \right\}$$

for those BC b_j that make sense in X.

$$D(A) := \Big\{ u \in W_p^8 \times W_p^7 \times W_p^7 \times W_p^4 : u \in X, \\ Au \in X, \quad b_j u = 0 \ \forall j \Big\}.$$

Theorem (D. 2009)

A generates analytic semigroup on X provided that $\mathcal{L} \supset \mathbb{C}_+$.

Application: viscous quantum hydrodynamics

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu_0 \triangle n, \\ \partial_t J - \operatorname{div}\left(\dfrac{J \otimes J}{n}\right) - \nabla(nT) + n\nabla V + \dfrac{\varepsilon^2}{2} n \nabla \dfrac{\triangle\sqrt{n}}{\sqrt{n}} \\ \qquad = \nu_0 \triangle J - \dfrac{1}{\tau} J + \mu \nabla n, \\ \partial_t(ne) - \operatorname{div}\left(\dfrac{J}{n}(ne + P)\right) + J \cdot \nabla V \\ \qquad = -\dfrac{2}{\tau}\left(ne - \dfrac{d}{2}n\right) + \nu_0 \triangle (ne) + \mu \operatorname{div} J, \\ \lambda_D^2 \triangle V = n - C(x) \end{cases}$$

$$P = nTI_d - \frac{\varepsilon^2}{4} n (\nabla \otimes \nabla) \ln n, \qquad ne = \frac{|J|^2}{2n} + \frac{d}{2} nT - \frac{\varepsilon^2}{8} n \triangle \ln n$$

Principal part in stationary case

$$\begin{pmatrix} \nu_0 \triangle & \mathrm{div} & 0 \\ -\frac{\varepsilon^2(d-1)}{4d} \nabla \triangle & \nu_0 \triangle I_d & 0 \\ -\frac{\varepsilon^2(d-1)}{4d} \frac{J}{n} \cdot \nabla \triangle & 0 & \nu_0 \triangle \end{pmatrix}$$

Stability of stationary states

isothermal case: $T \equiv \text{const} > 0$

$$
\begin{cases}
-\operatorname{div} J^* = \nu_0 \triangle n^*, \\
-\operatorname{div}\left(\dfrac{J^* \otimes J^*}{n^*}\right) - T\nabla n^* + n^*\nabla V^* + \dfrac{\varepsilon^2}{2} n^* \nabla \dfrac{\triangle \sqrt{n^*}}{\sqrt{n^*}} \\
\qquad\qquad = \nu_0 \triangle J^* - \dfrac{1}{\tau} J^*, \\
\lambda_D^2 \triangle V^* = n^* - C(x)
\end{cases}
$$

boundary conditions: $n = n_\Gamma, \quad \operatorname{div} J = 0, \quad J_\parallel = 0, \quad V = V_\Gamma$
$(n, J, V) =$ small perturbation of (n^*, J^*, V^*)

$$\overline{n} = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega C(x)\,\mathrm{d}x \qquad \text{constant reference value}$$

$$n\nabla V = \overline{n}\lambda_D^{-2} \nabla \triangle_D^{-1} (n - \overline{n}) + (\text{small})$$

stability relevant principal part

$$A(D) = \begin{pmatrix} \nu_0 \triangle & \text{div} \\ -\frac{\varepsilon^2}{4}\nabla\triangle + T\nabla - \overline{n}\lambda_D^{-2}\nabla\triangle_D^{-1} & \nu_0 \triangle - \frac{1}{\tau} \end{pmatrix}$$

eigenvalues of $\hat{A}(\xi)$ have real part $\leq -\frac{1}{2\tau}$, for all $\xi \in \mathbb{R}^d$ (for physically relevant values of λ_D, $\overline{n}$, τ)

Strategy

- forget Ω, take $\mathbb{R}^d$ instead
- diagonalize pseudodifferential symbol $\hat{A}(\xi)$ of $A(D)$:

  $$\hat{A}\hat{S} = \hat{S}\hat{\Lambda},$$

  columns of $\hat{S}$ unique up to scaling
- for $(\partial_t - \hat{A})\hat{U} = \hat{F}$, $\hat{W} := \hat{S}^{-1}\hat{U}$:

  $$\partial_t \hat{W} = \hat{\Lambda}\hat{W} + \hat{S}^{-1}\hat{F}.$$

  then good estimates for $\left\|\hat{W}\right\|_{L^2}^2 = \left\langle (\hat{S}^{-1})^*\hat{S}^{-1}\hat{U}, \hat{U} \right\rangle$
- determine scaling of $\hat{S}$ in such a way that $(\hat{S}^{-1})^*\hat{S}^{-1}$ makes sense in case of $x \in \Omega$ instead of $\mathbb{R}^d$

Result

Helmholtz decomposition: $L^p(\Omega;\mathbb{C}^d) = L^p_{\text{div}}(\Omega;\mathbb{C}^d) \oplus \nabla W^1_{p,0}(\Omega)$,
$J = J_{\text{div}} + \nabla\varphi_J$
Helmholtz projector commutes with $\triangle$
Ground space $\mathcal{H} = W^1_{2,0}(\Omega) \times L^2(\Omega;\mathbb{C}^d)$
Elements of $\mathcal{H}$: $\quad U = \binom{n}{J} = \binom{U_1}{\tilde{U}} = \binom{0}{\tilde{U}_{\text{div}}} + \binom{U_1}{\nabla\varphi_U}$
hand-crafted scalar product in $\mathcal{H}$:

$$\langle U, W\rangle_{\mathcal{H}} := \left\langle -\triangle_D^{-1}\left((2\tau)^{-1}U_1 + \nabla\varphi_U\right), (2\tau)^{-1}(W_1 + \triangle\varphi_W)\right\rangle_{L^2}$$
$$+ \left\langle(-C_1\triangle + T - C_2\triangle_D^{-1})U_1, W_1\right\rangle_{L^2} + \left\langle \tilde{U}_{\text{div}}, \tilde{W}_{\text{div}}\right\rangle_{L^2}$$

with $C_1 = \varepsilon^2/4$ and $C_2 = \lambda_D^{-2}\overline{n} - (4\tau^2)^{-1}$.

Main Result

Theorem (Chen, D. 2009)

If Ω is convex and $C_2 > (4\tau^2)^{-1}$ then

$$\Re\left\langle \left(A + \frac{1}{2\tau}\right) U, U \right\rangle_{\mathcal{H}} < 0, \qquad 0 \neq U \in \mathcal{H}.$$

Conclusions:

- $\sigma(A) \subset \{z : \Re z \leq -(2\tau)^{-1}\}$,
- semigroup generated by A decays like $e^{-t/(2\tau)}$ in $W_p^1(\Omega) \times L^p(\Omega; \mathbb{C}^d)$, ($p \geq 2$), by analyticity of the semigroup

Dissipativity in L^p (W.I.P.)

Theorem (Langer, Mazya 1999)

Higher order operators never dissipate in L^p for $p \neq 2$.

Example

$f(x) = 1 - x^4$ for $|x| < 1 - \varepsilon$, $f(x) = 0$ for $|x| > 1 - \varepsilon$ satisfies

$$\int_{-\infty}^{\infty} f''''(x)\overline{f(x)}|f(x)|^{p-2}\,\mathrm{d}x < 0$$

for $p > 3$ and $0 < \varepsilon \ll 1$. Hence $\triangle^2$ does not dissipate in $L^p(\mathbb{R}^1)$.

Dissipativity in mixed order Sobolev spaces (W.I.P.)

$$V := W_p^{m_1}(\Omega) \times W_p^{m_2}(\Omega) \times \ldots \times W_p^{m_N}(\Omega), \quad m_j \in \mathbb{N}_0,$$

$$\|f\|_V := \left(\int_\Omega \left(\sum_{j=1}^N \sum_{|\alpha|\le m_j} \mu_{j,\alpha} |\partial^\alpha f_j|^2 \right)^{p/2} \mathrm{d}x \right)^{1/p} =: \|M_f\|_{L^p}$$

with tuning parameters $\mu_{j,\alpha} > 0$.
Sub-differential of the norm: for $v \in V$,

$$J_V(v) := \left\{ \varphi \in V' \colon \ \|\varphi\|_{V'}^2 = \|v\|_V^2 = \langle \varphi, v \rangle \right\}$$

representation of $\varphi \in V'$:

$$\langle \varphi, u \rangle = \sum_{j=1}^N \sum_{|\alpha|\le m_j} \int_\Omega \varphi_{j,\alpha} \partial^\alpha u_j \, \mathrm{d}x$$

Dissipativity in mixed order Sobolev spaces (W.I.P.)

For $f \in V$, one element φ of $J_V(f)$ is given via

$$\varphi_{j,\alpha}(x) = c_0 \mu_{j,\alpha} \overline{\partial^\alpha f_j(x)} M_f(x)^{p-2}, \qquad c_0 := \|f\|_V^{2-p}.$$

A dissipates iff $\forall f \in V \; \exists \varphi \in J_V(f)$ with

$$\Re\langle Af, \varphi\rangle \leq 0.$$

Take

$$A := \begin{pmatrix} \nu_0 \triangle & \operatorname{div} \\ -\varepsilon^2 \nabla\triangle + T\nabla & \nu_0 \triangle \end{pmatrix}.$$

Theorem (D. 2011)

A dissipates in $W^1_p(\mathbb{R}^d) \times L^p(\mathbb{R}^d, \mathbb{R}^d)$ for small values of $(p-2)T$.