Existence and uniqueness of the solution of a supercritical free surface problem over an obstacle

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PLAN DE L'EXPOSE

- Introduction
- Position of the problem
- Transformation of the domain and the governing equations.
- Existence and uniqueness result

Introduction

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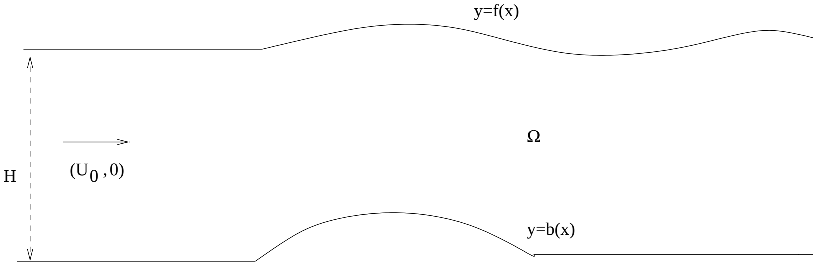

Fig1: The domain of the flow

Postion of the problem

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ / \ b(x) < \ y \ < H + \gamma(x)\}$$

Incompressible fluid , irrotational flow $\Rightarrow \exists \ \psi$ such that :

$$\begin{cases} \triangle\psi = 0 \quad\quad in \ \ \Omega \\ \psi = cste \quad\quad on \ \ y = b(x) \\ \psi = cste \quad\quad on \ \ y = H + \gamma(x) \\ \frac{\rho}{2}|\nabla\psi|^2 + \rho g y = C^{te} \ \ on \ \ y = H + \gamma(x) \\ \lim\limits_{x \to \pm\infty} \psi(x, y) = U_0 y \end{cases} \tag{1}$$

We adimensionalize $\Rightarrow F = \frac{U_0}{\sqrt{gH}}$

$$\Omega_b^\gamma = \{(x,y) \in \mathbb{R}^2 \ / \ b(x) < \ y \ < 1 + \gamma(x)\}$$

$$\begin{cases} \triangle\psi = 0 & \text{in } \Omega_b^\gamma \\ \psi = 0 & \text{on } y = b(x) \\ \psi = 1 & \text{on } y = 1 + \gamma(x) \\ \frac{F^2}{2}|\nabla\psi|^2 + y = \frac{F^2}{2} + 1 & \text{on } y = 1 + \gamma(x) \\ \lim\limits_{x \to \pm\infty} \psi(x,y) = y \end{cases} \tag{2}$$

Perturbation stream function $\Rightarrow \psi = y + \psi_p$

$$\begin{cases} \triangle\psi_p = 0 \quad in \;\; \Omega_b^\gamma \\ \psi_p = -b(x) \qquad on \;\; y = b(x) \\ \psi_p = -\gamma(x) \qquad on \;\; y = 1 + \gamma(x) \\ \\ \frac{F^2}{2}(|\nabla\psi_p|^2 + 2\frac{\partial\psi_p}{\partial y}) + \gamma(x) = 0 \;\; on \;\; y = 1 + \gamma(x). \\ \\ \lim\limits_{x\to\pm\infty} \psi_p(x, y) = 0 \end{cases} \tag{3}$$

Change of variables.

$$\begin{cases} \tilde{x} = x \\ \tilde{y} = \frac{y-b(x)}{1+\gamma(x)-b(x)} \end{cases} \qquad (4)$$

Ω_b^γ becomes

$$Q = \left\{(x,y) \in \mathbb{R}^2 / -\infty < x < +\infty,\ 0 < y < 1\right\}$$

Governing equations in the fix domain.

$$\begin{cases} \Delta\widetilde{\psi}_p + \mathcal{P}_b^{\gamma}\widetilde{\psi}_p = 0 \qquad in \;\; Q \\ \widetilde{\psi}_p(\widetilde{x},0) = -b(\widetilde{x}),\;\; \widetilde{x} \in \mathbb{R} \\ \widetilde{\psi}_p(\widetilde{x},1) = -\gamma(\widetilde{x}),\;\; \widetilde{x} \in \mathbb{R} \\ \\ \lim\limits_{\widetilde{x}\to\pm\infty} \widetilde{\psi}_p(\widetilde{x},\widetilde{y}) = 0 \\ \\ \frac{F^2}{2}[\left|\widetilde{\nabla}_{b,\gamma}\widetilde{\psi}_p\right|^2(\widetilde{x},1) + \frac{2}{1+\gamma-b}\frac{\partial\widetilde{\psi}_p}{\partial\widetilde{y}}(\widetilde{x},1)] + \gamma(\widetilde{x}) = 0 \end{cases} \qquad (5)$$

$$\widetilde{\nabla}_{b,\gamma} = \begin{pmatrix} \frac{\partial}{\partial \widetilde{x}} + \frac{-b' - \widetilde{y}(\gamma' - b')}{1+\gamma-b} \frac{\partial}{\partial \widetilde{y}} \\ \frac{1}{1+\gamma-b} \frac{\partial}{\partial \widetilde{y}} \end{pmatrix}$$

$$\mathcal{P}_b^{\gamma} = a_1 \frac{\partial^2}{\partial \widetilde{x} \partial \widetilde{y}} + a_2 \frac{\partial^2}{\partial \widetilde{y}^2} + a_3 \frac{\partial}{\partial \widetilde{y}}$$

$$a_1 = 2\frac{\widetilde{y}(b' - \gamma') - b'}{1 + \gamma - b}$$

$$a_2 = (\frac{a_1}{2})^2 - 1 + \frac{1}{(1 + \gamma - b)^2}$$

and

$$a_3 = \frac{-1}{1 + \gamma - b}[b'' + \widetilde{y}(\gamma'' - b'')] +$$

$$+ \frac{2}{(1 + \gamma - b)^2}(\gamma' - b')[b' + \widetilde{y}(\gamma' - b')]$$

The choice of the spaces

For $0 < \lambda < 1$ and $c > 0$ we put :

$$B_c^{m,\lambda}(\overline{Q}) = \left\{ v \in C^{m,\lambda}(\overline{Q}) / \sup_{k+l \le m} \sup_{(\tilde{x},\tilde{y}) \in \overline{Q}} e^{c\left|\tilde{x}\right|} \left| D^k_{\tilde{x}} D^l_{\tilde{y}} v(\tilde{x},\tilde{y}) \right| < \infty \right\}$$

$$\|v\|_{m,c,\lambda} = \sum_{k+l \le m} \sup_{(\tilde{x},\tilde{y}) \in \overline{Q}} e^{c\left|\tilde{x}\right|} \left| D^k_{\tilde{x}} D^l_{\tilde{y}} v(\tilde{x},\tilde{y}) \right| +$$

$$\sup_{k+l=m} \sup_{(\tilde{x},\tilde{y}) \ne (\tilde{x}',\tilde{y}')} \frac{\left| D^k_{\tilde{x}} D^l_{\tilde{y}} v(\tilde{x},\tilde{y}) - D^k_{\tilde{x}} D^l_{\tilde{y}} v(\tilde{x}',\tilde{y}') \right|}{\left[(\tilde{x}-\tilde{x}')^2 + (\tilde{y}-\tilde{y}')^2 \right]^{\lambda}}$$

Banach algebra.

$$B_c^{m,\lambda}(\mathbb{R}) = \left\{ v \in C^{m,\lambda}(\mathbb{R}) / \sum_{0\leq k\leq m} \sup_{x\in\mathbb{R}} e^{c|x|} \left| D_x^k v(x) \right| < \infty \right\}$$

$$\|v\|_{m,c,\lambda} = \sum_{0\leq k\leq m} \sup_{x\in\mathbb{R}} e^{c|x|} \left| D_x^k v \right| + \sup_{\substack{(x,x')\in\mathbb{R}^2 \\ x\neq x'}} \frac{|D_x^m v(x) - D_x^m v(x')|}{|x-x'|^{\lambda}}$$

Banach algebra.

b and γ in

$$B_c^{2,\lambda}(\mathbb{R}) = \left\{ v \in C^{2,\lambda}(\mathbb{R}) / \sum_{0 \leq k \leq 2} \sup_{x \in \mathbb{R}} e^{c|x|} \left| D_x^k v(x) \right| < \infty \right\}$$

$\overset{\sim}{\psi}$ in

$$B_c^{2,\lambda}(\overline{Q}) = \left\{ v \in C^{2,\lambda}(\overline{Q}) / \sup_{k+l \leq 2} \sup_{(\tilde{x}, \tilde{y}) \in \overline{Q}} e^{c\left|\tilde{x}\right|} \left| D_{\tilde{x}}^k D_{\tilde{y}}^l v(\tilde{x}, \tilde{y}) \right| < \infty \right\}$$

with $0 < \lambda < 1$ and $c > 0$

$$T : (b,\gamma) \in B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}) \mapsto T(b,\gamma) \in B_c^{1,\lambda}(\mathbb{R})$$

$$T(b,\gamma) = \frac{F^2}{2}\left[\left|\widetilde{\nabla}_{b,\gamma}\widetilde{\psi}\right|^2(\widetilde{x},1) + \frac{2}{1+\gamma-b}\frac{\partial\widetilde{\psi}}{\partial\widetilde{y}}(\widetilde{x},1)\right] + \gamma(\widetilde{x}) \tag{6}$$

Existence and uniqueness result

We have

$$T(0,0) = 0$$

We apply the implicit function theorem to the equation

$$T(b,\gamma) = 0$$

Differentiability of T with respect to b and γ

There exists an open ball $\mathcal{B}$ of radius r_0 centered at the origin of $B_c^{2,\lambda}(\mathbb{R})\times B_c^{2,\lambda}(\mathbb{R})$ such that T is continuously differentiable in this ball with respect to b and γ(see[1] and [5])

Let $h \in B_c^{2,\lambda}(\mathbb{R})$, we put $b = 0$ in $T(b,\gamma) = 0$, we derive with respect to γ in the direction h and we evaluate the derivative at $\gamma = 0$. We set

$$w_h = \frac{\partial\widetilde{\psi}}{\partial\gamma}\,|_{b=\gamma=0}\ .h$$

Expression of $\frac{\partial T}{\partial \gamma}(0,0).h$

The calculus gives :

$$\frac{\partial T}{\partial \gamma}(0,0).h = h + F^2 \frac{\partial w_h}{\partial y}(.,1)$$

where $w_h = \frac{\partial \widetilde{\psi}}{\partial \gamma}|_{b=\gamma=0}.\, h$ and verifies :

$$\left\{ \begin{array}{rcl} \Delta w_h & = & 0 \text{ in } Q \\ w_h(\widetilde{x},0) & = & 0,\ \widetilde{x} \in \mathbb{R} \\ w_h(\widetilde{x},1) & = & -h(\widetilde{x}),\ \widetilde{x} \in \mathbb{R} \end{array} \right. \tag{7}$$

Invertibility of $\frac{\partial T}{\partial \gamma}(0,0)$

The invertibility of the operator $\frac{\partial T}{\partial \gamma}(0,0)$ consists to prove that for a given q in $B_c^{1,\lambda}(\mathbb{R})$, there exists one and only one h in $B_c^{2,\lambda}(\mathbb{R})$ such that

$$h + F^2 \frac{\partial w_h}{\partial y}(.,1) = q \qquad (8)$$

with w_h verifying the system (7)

Theorem : The operator $h \mapsto h + F^2 \frac{\partial w_h}{\partial y}(.,1)$ is an isomorphism from $B_c^{2,\lambda}(\mathbb{R})$ into $B_c^{1,\lambda}(\mathbb{R})$, w_h is the solution of the system (7)

Result deduced from :

Proposition Consider the boundary value problem

$$\left\{ \begin{array}{rcl} \Delta u & = & 0 \ \text{ in } Q \\ u(x,0) & = & 0, \ x \in \mathbb{R} \\ -u(x,1) + F^2 \frac{\partial u}{\partial y}(x,1) & = & q(x), \ x \in \mathbb{R} \end{array} \right. \qquad (9)$$

with assumption that $q \in B_c^{1,\lambda}(\mathbb{R})$; for $F > 1$, the problem (9) has a unique solution u in $B_c^{2,\lambda}(\overline{Q})$.

$$V = \{v \in H^1(Q)/v(x,0) = 0\} \tag{10}$$

The variational formulation of the problem (9) is :

$$\textit{Find } u \in V \textit{ such that } a(u,v) = l(v) \; \forall \, v \in V \tag{11}$$

where

$$a(u,v) = \int_Q \nabla u \nabla v \, dxdy - \frac{1}{F^2} \int_{\Gamma_u} u \, v \, d\sigma$$

$$l(v) = \frac{1}{F^2} \int_{\Gamma_u} qv \, d\sigma$$

Γ_u is the upper bound of Q.

We verify that the bilinear form $a(.,.)$ is continue and coercive on $V \times V$; The linear form "l" is continuous on V, then the Lax-Milgram theorem implies that the problem (9) has a unique solution u in V.

From :

$$-u(x,1) + F^2 \frac{\partial u}{\partial y}(x,1) = q(x)$$

$q \in B_c^{1,\lambda}(\mathbb{R}) \subset H^1(\mathbb{R})$ and $u(.,1) \in H^{1/2}(\mathbb{R})$

We deduce that :

$$\frac{\partial u}{\partial y}(x,1) \in H^{1/2}(\mathbb{R})$$

Then $u \in H^2(Q)$

The weak solution u is in $C^{2,\lambda}(\overline{Q})$

$u \in C^{2,\lambda}(\overline{Q})$ (see theorems 6.31 and 6.15 Gilbard-Trudinger)

u is in $B_c^{2,\lambda}(\overline{Q})$

It remains to verify

$$\sup_{k+l\leq 2} \sup_{\overline{Q}} e^{c|x|} \left| D_x^k D_y^l u \right| < \infty \tag{12}$$

which is equivalent to

$$\sup_{\overline{Q}} e^{c|x|} \left| D_x^k D_y^l u \right| < \infty \quad \forall k, l \in \mathbb{N} \ ; \ k + l \leq 2 \tag{13}$$

Let

$$v = u + \frac{y}{F^2}\int_{x+y-1}^{x} q(s)ds \qquad (14)$$

where u is the solution of the problem(9). Then v is the solution of the following homogeneous problem

$$\left\{\begin{array}{rcl} -\Delta v &=& f \ \text{ in } Q \\ v(x,0) &=& 0, \ x \in \mathbb{R} \\ -v(x,1) + F^2\frac{\partial v}{\partial y}(x,1) &=& 0, \ x \in \mathbb{R} \end{array}\right. \qquad (15)$$

where $f = -\triangle(\frac{y}{F^2}\int_{x+y-1}^{x} q(s)ds)$ is in $B_c^{0,\lambda}(\overline{Q})$ because the function $\frac{y}{F^2}\int_{x+y-1}^{x} q(s)ds$ is in $B_c^{2,\lambda}(\overline{Q})$

Introduce the Green's function g associated to the problem (15). Because of the structure of the problem, $g(x, y; x', y')$ can be written as $G(x - x', y, y')$ where G is

$$\Delta G = -\delta(x - x')\delta(y - y') \quad in\ Q$$

$$G(x - x', 0, y') = 0,$$

$$-G(x - x', 1, y') + F^2 \frac{\partial G}{\partial y}(x - x', 1, y') = 0,$$

$$|G| < \infty \ \ for\ |x| \to \infty$$

Denote by D^k the differential operator in the variables x and y of order k. We prove that, there exist two strictly positive constants C and $\widetilde{c}$ such that

$$|D^k G(x - x', y, y')| \leq C e^{-\widetilde{c}|x-x'|} \qquad (16)$$

for $|x - x'| \geq 1$.

G enjoys the following properties :
There exist two strictly positives constants K and k such that

$$|G(x-x',y,y')| \leq K|Log[|x-x'|+|y-y'|]|+k \qquad (17)$$

$$|DG(x-x',y,y')| \leq K[|x-x'|+|y-y'|]^{-1} \qquad (18)$$

$$|D^2G(x-x',y,y')| \leq K[|x-x'|+|y-y'|]^{-2} \qquad (19)$$

whenever $|x-x'| \leq 1$

Now, to prove (13) for v, use the Green's function to write

$$v(x,y) = \int_Q G(x-x', y, y') f(x', y') dx' dy' \tag{20}$$

which leads to :

$$\sup_{\overline{Q}} e^{c|x|} |v(x,y)| < \infty$$

We proceed in exactly the same fashion to obtain an estimate for the weighted norm of the first derivatives of v because differentiation under the integral sign is legitimate since the first derivatives have integrable singularities at $(x, y) = (x', y')$.

To estimate the weighted norm of the second derivatives of v, consider $w = \omega v$ where
$\omega \in C^{\infty}(\mathbb{R})$, $\omega \geq \omega_0$ and $\omega = exp c|x|$ for $|x| > 1$.
w and the first derivatives of w are already known to be bounded so we use classical Hölder estimates for elliptic equations to conclude.

We have established that $v \in B_c^{2,\lambda}(\overline{Q})$, so from (14), $u \in B_c^{2,\lambda}(\overline{Q})$.
Then, for q given in $B_c^{1,\lambda}(\mathbb{R})$, we conclude that the problem :

$$\begin{cases} \Delta u = 0 \text{ in } Q \\ u(x,0) = 0, \; x \in \mathbb{R} \\ -u(x,1) + F^2 \frac{\partial u}{\partial y}(x,1) = q(x), \; x \in \mathbb{R} \end{cases} \tag{21}$$

has a unique solution u in $B_c^{2,\lambda}(\overline{Q})$ and $h = -u(x,1)$ is the solution of

$$h + F^2 \frac{\partial w_h}{\partial y}(.,1) = q$$

so $\frac{\partial T}{\partial \gamma}(0,0)$ is invertible.

We give our existence and uniqueness result :

Theorem : There exists $\widetilde{c} > 0$ such that whenever $0 < c < \widetilde{c}$, there exist an open ball $\mathcal{B}$ of radius r_0 centered at the origin of $B_c^{2,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$, $0 < \lambda < 1$, an open neighborhood $\mathcal{V}_b$ of zero in $B_c^{2,\lambda}(\mathbb{R})$, and a mapping $\mathrm{g} : \mathcal{V}_b \longrightarrow B_c^{2,\lambda}(\mathbb{R})$ of class $\mathcal{C}^1$, such that the following equivalence holds :
$\{\forall (b,\gamma) \in \mathcal{B},\, T(b,\gamma) = 0\} \Leftrightarrow \{b \in \mathcal{V}_b, \gamma = \mathrm{g}(b)\}$.

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