On Korn's inequality on Orlicz spaces

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Korn's inequality

Let $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field.

$$\nabla \mathbf{v} := \texttt{gradient},$$
$$\varepsilon(\mathbf{v}) := \texttt{symmetric gradient} = \frac{1}{2}\big((\nabla \mathbf{v}) + (\nabla \mathbf{v})^T\big).$$

Observation: Some PDE give control on $\varepsilon(\mathbf{v})$ rather than $\nabla \mathbf{v}$.

Note that $|\varepsilon(\mathbf{v})| \leq |\nabla \mathbf{v}|$ but $|\nabla \mathbf{v}| \not\leq c\,|\varepsilon(\mathbf{v})|$.

Matter of taste: We like Sobolev spaces using $\nabla \mathbf{v}$.

Objective of Korn's inequality:

Control global information on $\nabla \mathbf{v}$ by the one of $\varepsilon(\mathbf{v})$

Objectivity

In fluid dynamics we deal with equations like

$$\partial_t \mathbf{v} + [\nabla \mathbf{v}]\mathbf{v} + \operatorname{div} \mathbf{T} + \nabla q, = \mathbf{f},$$
$$\operatorname{div} \mathbf{v} = 0$$

with velocity $\mathbf{v}$, pressure q, stress tensor $\mathbf{T}$ (material properties).

Material objectivity

The response of a material is independent of the particular observer.

First observer at position x
Second observer at position $\widetilde{x} = \mathbf{Q}(t)x + y_0(t)$, with $\mathbf{Q}(t)$ orthonormal.

Objectivity requires: $\widetilde{\mathbf{T}} = \mathbf{Q}(t)\,\mathbf{T}\,\mathbf{Q}(t)^T$

Why symmetric gradients?

The Eulerian gradient $\mathbf{L} := (\nabla \mathbf{v})^T$ satisfies

$$\widetilde{\mathbf{L}} = \mathbf{Q}(t)\mathbf{L}\mathbf{Q}(t)^T + \mathbf{Q}(t)\dot{\mathbf{Q}}(t)^T$$

and is NOT objective.

The symmetric gradient $\varepsilon := \varepsilon(\mathbf{v}) = \frac{1}{2}((\nabla \mathbf{v}) + (\nabla \mathbf{v})^T)$ satisfies

$$\widetilde{\varepsilon} = \frac{1}{2}(\widetilde{\mathbf{L}} + \widetilde{\mathbf{L}}^T) = \mathbf{Q}(t)\,\varepsilon\,\mathbf{Q}(t)^T + \frac{d}{dt}\big(\mathbf{Q}(t)\mathbf{Q}(t)^T\big) = \mathbf{Q}(t)\,\varepsilon\,\mathbf{Q}(t)^T$$

and is objective.

Conclusion: $\mathbf{T}$ may not depend on $\nabla \mathbf{v}$ but only on $\varepsilon(\mathbf{v})$.

Fluid dynamics (and Elasticity)

Power law fluids: $\mathbf{T}(\varepsilon) = |\varepsilon|^{p-2}\varepsilon$ (blood, ketchup)

Gives control on $\mathbf{T}(\varepsilon(\mathbf{v})) : \varepsilon(\mathbf{v}) = |\varepsilon(\mathbf{v})|^p$

PDE gives $\varepsilon(\mathbf{v}) \in L^p$, but we prefer $\nabla \mathbf{v} \in L^p$

More general fluids: $\mathbf{T}(\varepsilon) = \frac{\varphi'(|\varepsilon|)}{|\varepsilon|}\varepsilon$ for Orlicz function φ.

Gives control on $\mathbf{T}(\varepsilon(\mathbf{v})) : \varepsilon(\mathbf{v}) \approx \varphi(|\varepsilon(\mathbf{v})|)$

PDE gives $\varepsilon(\mathbf{v}) \in L^\varphi$, but we prefer $\nabla \mathbf{v} \in L^\varphi$

Prandtl-Eyring fluids: $\varphi(t) \approx t\,\ln(1+t)$ (almost perfect plasticity)

PDE gives $\varepsilon(\mathbf{v}) \in L^{t \ln t}$, but we prefer $\nabla \mathbf{v} \in L^{t \ln t}$

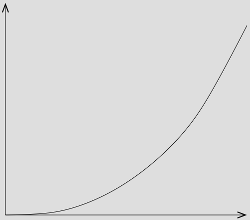
Orlicz spaces

We say $\varphi : [0,\infty) \to [0,\infty]$ is an Orlicz*-function if

- φ is convex and left-continuous
- $\varphi(0) = 0$, $\lim_{t\to 0} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

The Orlicz space $L^\varphi := \left\{ f : \lim_{\lambda\to\infty} \int \varphi(|f|/\lambda)\, dx = 0 \right\}$ is a Banach space

with $\|f\|_\varphi := \inf \left\{ \lambda > 0 : \int \varphi(\lambda|f/\lambda|)\, dx \leq 1 \right\}$



Orlicz function	Orlicz space
$\varphi(t) := t$	$L^\varphi = L^1$
$\varphi(t) := t^p$	$L^\varphi = L^p$
$\varphi(t) := \infty\chi_{(1,\infty)}$	$L^\varphi = L^\infty$
$\varphi(t) := t\,\ln(1+t)$	$L^\varphi = L^{t \ln t}$

Rigid motions

To control $\|\nabla \mathbf{v}\|_\varphi$ by $\|\varepsilon(\mathbf{v})\|_\varphi$ we have to ensure that

$$\varepsilon(\mathbf{v}) = 0 \qquad \text{implies} \qquad \nabla \mathbf{v} = 0.$$

The kernel of ε is bigger:

$$\begin{aligned} \ker(\nabla) &= \{\mathbf{v}(x) = \mathbf{b} \,:\, \mathbf{b} \in \mathbb{R}^n\}, \\ \ker(\varepsilon) &= \{\mathbf{v}(x) = \mathbf{A}x + \mathbf{b} \,:\, \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{b} \in \mathbb{R}^n \text{ with } \mathbf{A} + \mathbf{A}^T = 0\} \\ &=: \texttt{(infinitesimal generator of) rigid motions} \end{aligned}$$

To fix this problem, we need one of the following extra conditions

(a) boundary conditions; e.g. $\mathbf{v} = 0$ on $\partial\Omega$ or at infinity

(b) quotient spaces: e.g. $\|\nabla \mathbf{v} - \langle \nabla \mathbf{v}\rangle\|_\varphi$ by $\|\varepsilon(\mathbf{v}) - \langle \varepsilon(\mathbf{v})\rangle\|_\varphi$

Note that (b) implies (a).

The quadratic case

Let $\mathbf{v} \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then by partial integration

$$\|\varepsilon(\mathbf{v})\|_2^2 = \langle \varepsilon(\mathbf{v}), \varepsilon(\mathbf{v}) \rangle = \frac{1}{2}\|\nabla \mathbf{v}\|_2^2 + \frac{1}{2}\|\operatorname{div} \mathbf{v}\|_2^2.$$

Theorem

For $\mathbf{v} \in W^{1,2}(\mathbb{R}^n)$ *holds* $\|\nabla \mathbf{v}\|_2 \leq \sqrt{2}\|\varepsilon(\mathbf{v})\|_2$.

Alternative: Fourier symbols

Norm vs. modular

We have two types of boundedness for $T : L^\varphi \to L^\varphi$ linear:

Norm: $$\|Tf\|_\varphi \leq c\,\|f\|_\varphi$$

Modular: $$\int \varphi(|Tf|)\,dx \leq \int \varphi(c\,|f|)\,dx.$$

Theorem

Modular boundedness implies norm boundedness.

Proof: Use $\{\|f\|_\varphi \leq 1\} = \{\int \varphi(|f|)\,dx \leq 1\}$.

More on Orlicz spaces

We say $\varphi \in \Delta_2$ if: $\varphi(2t) \le c\,\varphi(t)$ for all $t \ge 0$.

Conjugate Orlicz function: $\varphi^*(u) := \sup_{u \ge 0} \big(t\,u - \varphi(t)\big)$

We say $\varphi \in \nabla_2$ if $\varphi^* \in \Delta_2$.

$\varphi(t)$	L^φ	L^φ	$\varphi \in \Delta_2$	$\varphi \in \nabla_2$
t	L^1	L^∞	YES	NO
t^p, $1 < p < \infty$	L^p	$L^{p'}$	YES	YES
$\infty\chi_{(1,\infty)}$	L^∞	L^1	NO	YES
$t\,\ln(1+t)$	$L^{t \ln t}$	$L^{\exp}$	YES	NO

Main result

Theorem [Breit, Diening '11]: The following are equivalent (B is a ball)

(a) $\varphi \in \Delta_2 \cap \nabla_2$.

(b) $\displaystyle\int_B \varphi(|\nabla \mathbf{v} - \langle \nabla \mathbf{v}_B \rangle|)\,dx \leq \int_B \varphi(A_1\,|\varepsilon(\mathbf{v}) - \langle \varepsilon(\mathbf{v})\rangle_B|)\,dx.$

(c) $\|\nabla \mathbf{v} - \langle \nabla \mathbf{v}\rangle_B\|_{L^\varphi(B)} \leq A_2 \|\varepsilon(\mathbf{v}) - \langle \varepsilon(\mathbf{v})\rangle_B\|_{L^\varphi(B)}.$

(d) $\displaystyle\int_{\mathbb{R}^n} \varphi(|\nabla \mathbf{v}|)\,dx \leq \int_{\mathbb{R}^n} \varphi(A_3\,|\varepsilon(\mathbf{v})|)\,dx.$

(e) $\|\nabla \mathbf{v}\|_{L^\varphi(\mathbb{R}^n)} \leq A_4 \|\varepsilon(\mathbf{v})\|_{L^\varphi(\mathbb{R}^n)}.$

Trivial: (b) $\Rightarrow$ (c) $\Rightarrow$ (e) and (b) $\Rightarrow$ (d) $\Rightarrow$ (e)

Remains only: (a) $\Rightarrow$ (b) and (e) $\Rightarrow$ (a)

Singular integral representation

Representation by Reshneyak

If $\mathbf{v} \in C^\infty(\overline{\Omega})$ with Ω convex, then

$$\nabla \mathbf{v} = T(\varepsilon(\mathbf{v})) + Q(\varepsilon(\mathbf{v})),$$

where T is singular integral operator and Q is multiplication operator with smooth function.

Observation:
Singular integral operators behave nicely on L^φ if and only if $\varphi \in \Delta_2 \cap \nabla_2$.

Conjecture:
Same conditions for Korn's inequality.

Via derivatives

It holds $\partial_i \partial_j v_k = \partial_i \varepsilon_{jk}(\mathbf{v}) + \partial_j \varepsilon_{ki}(\mathbf{v}) - \partial_k \varepsilon_{ij}(\mathbf{v})$.

Negative norm theorem [Nečas]:

For $1 < p < \infty$, $W^{-1,p}(B) := (W_0^{1,p'}(B))^*$ holds $\|\nabla f\|_{-1,p} \approx \|f - \langle f \rangle\|_p$.

Proof of Korn's inequality

$$\|\nabla \mathbf{v} - \langle \nabla \mathbf{v} \rangle\|_{L^p(B)} \approx \|\nabla\nabla \mathbf{v}\|_{-1,p} \approx \|\nabla \varepsilon(\mathbf{v})\|_{-1,p} \approx \|\varepsilon(\mathbf{v}) - \langle \varepsilon(\mathbf{v}) \rangle\|_{L^p(B)}$$

[Diening, Schumacher, Růžička '11]:

Same for $L^p_w(B)$ with $w \in A_p$ Muckenhoupt weights, $1 < p < \infty$.

L^φ estimates by extrapolation or interpolation; $\varphi \in \Delta_2 \cap \nabla_2$.

Works also for John domains.

Via sharp functions

For $s > 1$ define $M^\sharp_s f(x) := \sup\limits_{B \ni x} \left(\fint_B |f - \langle f \rangle|^s \, dx \right)^{\frac{1}{s}}$.

By L^s theory: $M^\sharp_s(\nabla \mathbf{v}) \leq c_s M^\sharp_s(\varepsilon(\mathbf{v}))$.

Now use Fefferman-Stein for L^φ: $\|g - \langle g \rangle\|_\varphi \approx \|M^\sharp_s g\|_\varphi$.

Requires $\varphi \in \Delta_2 \cap \nabla_2$ and $\mathrm{index}(\varphi) > s$.

Proof of Korn's inequality:

$$\|\nabla \mathbf{v} - \langle \nabla \mathbf{v} \rangle\|_\varphi \approx \|M^\sharp_s(\nabla \mathbf{v})\|_\varphi \approx \|M^\sharp_s(\varepsilon(\mathbf{v}))\|_\varphi \approx \|\varepsilon(\mathbf{v}) - \langle \varepsilon(\mathbf{v}) \rangle\|_\varphi$$

Proved: $\varphi \in \Delta_2 \cap \nabla_2$ implies Korn.

Korn implies Δ_2-condition (1/2)

Theorem

Korn fails on L^∞, i.e. $\|\nabla \mathbf{v}\|_\infty \not\lesssim c\,\|\varepsilon(\mathbf{v})\|_\infty$ for $\mathbf{v} \in C_0^\infty(B_1)$.

Define $\mathbf{v}(x) := \mathbf{Q}x \ln|x|$ with $\mathbf{Q}$ antisymmetric. Then $\mathbf{v} \in W_0^{1,1}(B_1)$ and

$$\nabla \mathbf{v} = \frac{x \otimes \mathbf{Q}x}{|x|^2} + \mathbf{Q}\ln|x| \qquad \in \mathrm{BMO} \setminus L^\infty$$

$$\varepsilon(\mathbf{v}) = \frac{x \otimes \mathbf{Q}x}{|x|^2} \qquad \in L^\infty.$$

Korn fails implies Δ_2-condition (2/2)

Theorem

Korn implies $\varphi \in \Delta_2$.

Scale $\mathbf{v} := \mathbf{Q}x \ln |x|$ to ball B, use $|\mathbf{Q}| = t$ and Korn to show

$$|B|\varphi(t) = 1 \qquad \Rightarrow \qquad |B|\varphi(2t) \leq K.$$

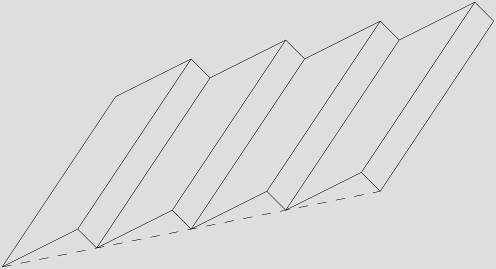
In particular,

$$\varphi(2t) \leq K\,\varphi(t).$$

Korn implies ∇_2 -condition (1/3)

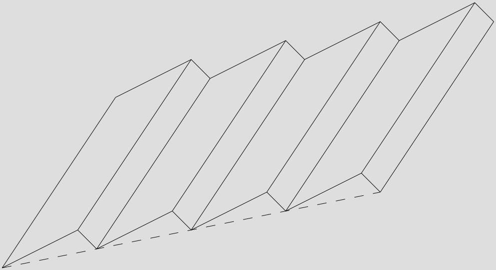
Based on Conti-Faraco-Maggi '05 (failure of Korn on L^1)
using laminates by Müller et al '99-

Construct functions $\mathbf{v}$ like:



Oscillating, locally affine linear (locally constant gradients)

Korn implies ∇_2 -condition (2/3)



Gradient: $\nabla \mathbf{v} = \mathbf{G}$ or $\nabla \mathbf{v} = \mathbf{H}$ with $\operatorname{rank}(\mathbf{G} - \mathbf{H}) = 1$

1st order laminate: $\nu := (1-\lambda)\delta_{\mathbf{G}} + \lambda \delta_{\mathbf{H}}$, $\quad \operatorname{avg}(\nu) := (1-\lambda)\mathbf{G} + \lambda \mathbf{H}$

Replace flat pieces by more oscillations:
2nd order laminate: Replace $\delta_{\mathbf{G}}$ by 1st order laminate with $\operatorname{avg}(\nu) = \mathbf{G}$

Korn implies ∇_2 -condition (3/3)

Theorem

For a laminate ν with average $\mathbf{C}$ *exist* $\mathbf{v}_i$ *with boundary data* $\mathbf{C}x$ *and*

$$\fint_{B_r} \varphi(\nabla \mathbf{v}_i)\,dx \to \int_{\mathbb{R}^{n\times n}} \varphi(F)\,d\nu(F), \qquad \fint_{B_r} \varphi(\varepsilon(\mathbf{v}_i))\,dx \to \int_{\mathbb{R}^{n\times n}} \varphi(F^{\mathrm{sym}})\,d\nu(F),$$

Construct laminate $\nu^{(n)} = 2^{-n}\delta_{\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}} + \sum_j \lambda_j \delta_{\mathbf{H}_j}$

with $\mathbf{G}$ symmetric and $\mathbf{H}_j$ antisymmetric and $\mathrm{avg}(\nu^{(n)}) = \begin{pmatrix} 0 & 2^{-n}t \\ 2^{-n}t & 0 \end{pmatrix}$

$\mathbf{H}_j$ not seen in $\varepsilon(\mathbf{v}_j)$! Korn fails on L^1

2^{-n} moves inside of φ! Korn implies ∇_2

Summary

Theorem [Breit, Diening '11]: (short version)

Korn on L^φ is equivalent to $\varphi \in \Delta_2 \cap \nabla_2$.

So we cannot use Korn for Prantl-Eyring fluids, where $\varepsilon(\mathbf{v}) \in L^{t \ln t}$.

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Summary

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