# Non-smooth atomic decompositions in function spaces

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## Conference FSDONA 2011

September 18-24, 2011

[1]joint work with Jan Vybíral

# Outline

Introduction

Besov spaces
- Classical approach
- Smooth atomic decompositions
- Non-smooth atomic decompositions

Applications
- Traces on Lipschitz domains
- Pointwise multipliers

# Outline

# Decomposition of function spaces

**Idea:**

represent functions as linear combination of basic functions

$$f \text{ in function space } X \qquad \Longleftrightarrow \qquad f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}$$

where

$\lambda_{jm}$ ... coefficients in certain sequence space

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- **Authors:**
  Calderón, Coifman, Daubechies, DeVore, Fefferman, Frazier, Jawerth, Meyer, Popov, Rochberg, Wilson, ...

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  $\rightsquigarrow$ Besov spaces with $p = q$

- [Schneider & Vybíral, 2011]: classical Besov spaces $\mathbf{B}^s_{p,q}$

  $\rightsquigarrow$ now also $p \neq q$

  $\curvearrowright$ applications to pointwise multipliers, traces on Lipschitz domains

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# Classical approach

Higher order differences

$$\Delta_h^1 f(x) = f(x+h) - f(x), \qquad (\Delta_h^{r+1} f)(x) = \Delta_h^1(\Delta_h^r f)(x), \quad r \in \mathbb{N}, \;\; h \in \mathbb{R}^n$$

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Besov spaces $0 < p, q \leq \infty$, $s > 0$

$$\|f|\mathbf{B}_{p,q}^s(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \left( \int_0^\infty [t^{-s}\omega_r(f,t)_p]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, \qquad r > s$$

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- Hölder-Zygmund spaces

$$\mathcal{C}^s = \mathbf{B}^s_{\infty,\infty}, \qquad s > 0$$

# Atomic characterization

## Definition (Smooth atoms)

Let $K \in \mathbb{N}_0$ and $c > 1$. A function $a \in C^K(\mathbb{R}^n)$ is called a $K$-atom if

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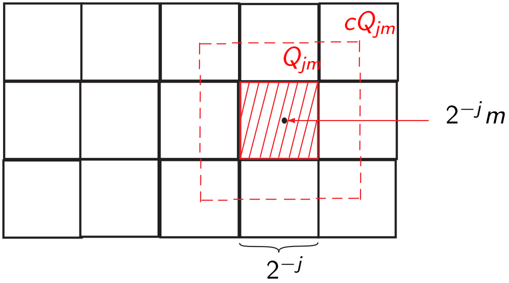

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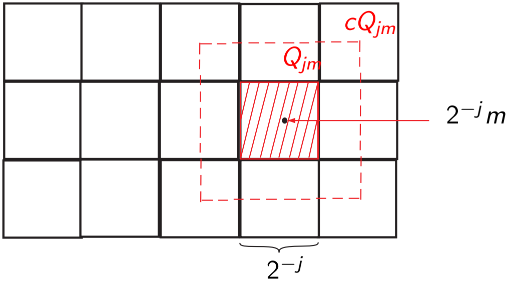

## Theorem (Hedberg, Netrusov, 2006)

*Let* $0 < p, q \leq \infty$, $s > 0$, *and*

$$K \geq (1 + [s])$$

*be fixed. Then* $f \in \mathbf{B}^s_{p,q}$ *if, and only if,*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}, \qquad \lambda \in b^s_{p,q},$$

*where the* $a_{jm}$ *are* $K$*-atoms, convergence being in* $L_p(\mathbb{R}^n)$*. Furthermore*

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- $$\|\lambda|b^s_{p,q}\| := \left( \sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \right)^{q/p} \right)^{1/q}$$

# Non-smooth atoms

compact support of smooth atoms yields

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Let $\sigma > 0$, $0 < p \leq \infty$ and $c > 1$. A function $a \in \mathbf{B}^\sigma_{p,p}$ is called a $(\sigma, p)$-atom if

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- if $\sigma < \frac{n}{p}$ then $(\sigma, p)$-atoms might be unbounded

# Non-smooth atomic decomposition

**Theorem (S., Vybíral, 2011)**

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- $\mathrm{Lip}$-atoms (if $0 < s < 1$)

$$a \in \mathrm{Lip}(\mathbb{R}^n) \qquad \text{with} \qquad \|a(2^{-j}\cdot)|\mathrm{Lip}(\mathbb{R}^n)\| \leq 1$$

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# Traces on Lipschitz domains

**Theorem (S., Vybíral, 2011)**

*Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary $\Gamma$. Then for $0 < s < 1$ and $0 < p, q \leq \infty$,*

$$\mathrm{Tr}\ \mathbf{B}^{s+\frac{1}{p}}_{p,q}(\Omega) = \mathbf{B}^{s}_{p,q}(\Gamma).$$

- no more restrictions on the parameters $s$ and $p$!

# Pointwise multipliers

- connection between pointwise multipliers for $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and selfsimilar spaces

**Theorem (S., Vybíral, 2011)**

*Let* $s > 0$ *and* $0 < p, q \leq \infty$. *Then*

(i)

$$\bigcup_{\sigma > s} \mathbf{B}^{\sigma}_{p,q,\mathrm{selfs}}(\mathbb{R}^n) \subset M(\mathbf{B}^s_{p,q}(\mathbb{R}^n)) \hookrightarrow \mathbf{B}^s_{p,q,\mathrm{selfs}}(\mathbb{R}^n)$$

(ii) *Additionally, if* $0 < p \leq 1$,

$$M(\mathbf{B}^s_{p,p}(\mathbb{R}^n)) = \mathbf{B}^s_{p,p,\mathrm{selfs}}(\mathbb{R}^n).$$

↷ study characteristic functions $\chi_\Omega$ as multipliers

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