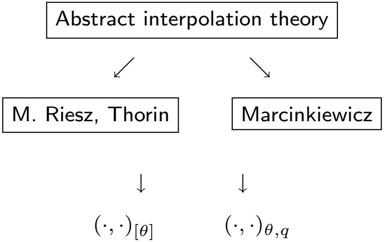
Limiting interpolation methods

Fernando Cobos

Universidad Complutense de Madrid

FSDONA 2011


Abstract interpolation theory
M. Riesz, Thorin
Marcinkiewicz
$(\cdot,\cdot)_{[\theta]}$
$(\cdot,\cdot)_{\theta,q}$

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▷ J. Bergh and J. Löfström, Springer, 1976.

▷ C. Bennett and R. Sharpley, Academic Press, 1988.

▷ Yu. A. Brudnyi and N. Ya. Krugljak, North-Holland, 1991.

▷ W.O. Amrein, A. Boutet de Monvel and V. Georgescu, Birkhäuser, 1996.

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Peetre's K- and J-functional:

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \; : a = a_0 + a_1,\ a_j \in A_j\},\ t > 0,\ a \in A_0 + A_1,$$

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For $0 < \theta < 1$ and $1 \le q \le \infty$

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$$= \left\{ a \in A_0 + A_1 : a = \int_0^\infty u(t)\, \frac{dt}{t} \text{ with } \left(\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

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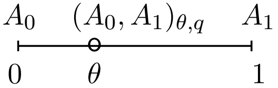

• If $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ are Banach couples and $T \in \mathcal{L}(A_0 + A_1, B_0 + B_1)$ with $T : A_j \longrightarrow B_j$ bounded for $j = 0, 1$ $\quad [T \in \mathcal{L}(\bar{A}, \bar{B})]$ then

$$T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q} \text{ is bounded and}$$

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Example.- Let (Ω, μ) be any σ-finite measure space. Put $A_0 = L_1$ and $A_1 = L_\infty$.

$$K(t, f) = \int_0^t f^*(s)ds$$

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Then for $1/p = 1 - \theta$ and $1 \le q \le \infty$, we have

$$(L_1, L_\infty)_{\theta,p} = L_p\,, \quad (L_1, L_\infty)_{\theta,q} = L_{p,q}\,,$$

$$L_{p,q} = \left\{ f : \|f\|_{L_{p,q}} = \left(\int_0^\infty (t^{\frac{1}{p}-1} \int_0^t f^*(s)ds)^q \,\frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

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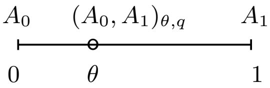

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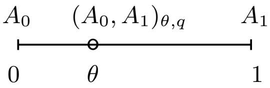

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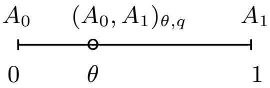

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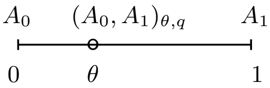

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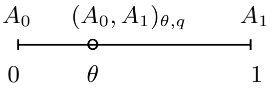

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The reason for cutting the integral is

$$K(t,a) \sim t\|a\|_{A_1} \quad \text{for} \quad 0 < t \leq 1.$$

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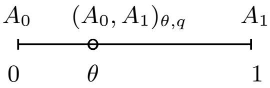

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$$\text{Moreover} \quad \|a\|_{\theta,q} \sim \left(\int_1^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t} \right)^{1/q}$$

▷ M.E. Gomez and M. Milman, J. London Math. Soc. 34 (1986) 305-316.

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The case of J-spaces with $\theta = 0$

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$\triangleright$ F. Cobos, L.M. Fernández-Cabrera, T. Kühn and T. Ullrich, J. Funct. Anal. 256 (2009) 2321-2366.

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For $0 < \theta < 1$ and $A_0 \hookrightarrow A_1$

$$\|a\|_{\theta,q} \sim \inf \Big\{ \Big(\int_1^\infty [t^{-\theta} J(t, u(t))]^q \frac{dt}{t} \Big)^{1/q} : a = \int_1^\infty u(t) \frac{dt}{t} \Big\}.$$

Definition. Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \leq q \leq \infty$. The space $(A_0, A_1)_{0,q;J}$ is formed by all those elements $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that

$$a = \int_1^\infty u(t) \frac{dt}{t} \quad \text{(convergence in } A_1\text{)} \tag{1}$$

and

$$\Big(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \Big)^{1/q} < \infty \tag{2}$$

(with the usual modification if $q = \infty$). We set

$$\|a\|_{0,q;J} = \inf \Big\{ \Big(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \Big)^{1/q} \Big\}$$

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- If $q = 1$ then $(A_0, A_1)_{0,1;J} = A_0$.

This construction produces spaces which are very close to A_0.

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$$(L_{p_0,q}, L_{p_1,q})_{0,q;J} = L_{p_0,q}(\log L)_{-1/q'} =$$

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THEOREM.- Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$. Suppose that $T \in \mathcal{L}(\bar{A}, \bar{B})$. Then for $1 < q \le \infty$ we have

$$\|T\|_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,q;J}} \le C \|T\|_{A_0,B_0} \left[1 + (\log \frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}})_+ \right].$$

x_+ stands for $\max\{x, 0\}$.

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THEOREM.- Let A_0,A_1 be Banach spaces with $A_0\hookrightarrow A_1$ and A_0 dense in A_1. Assume that $1<q<\infty$ and let $1/q+1/q'=1$. Then we have

$$(A_0,A_1)'_{0,q;J}=(A_1',A_0')_{1,q';K}$$

and

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Compactness

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Counterexample.

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$$I : (\ell_q, \ell_q(2^{-n}))_{0,q;J} = \ell_q(n^{-1/q'}) \longrightarrow (\ell_q, \ell_q(e^{-n}))_{0,q;J} = \ell_q(n^{-1/q'}) \text{ is not compact.}$$

THEOREM.- Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$, $B_0 \hookrightarrow B_1$, let $1 \leq q \leq \infty$ and $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T : A_0 \longrightarrow B_0$ is compact, then

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▷ F. Cobos, L.M. Fernández-Cabrera, T. Kühn and T. Ullrich, J. Funct. Anal. 256 (2009) 2321-2366.

- Estimates for entropy numbers of interpolated operators when $B_0 = B_1$.

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$\triangleright$ F. Cobos, L.M. Fernández-Cabrera, and M. Mastylo, J. London Math. Soc. 82 (2010) 501-525.

• Necessary and sufficient conditions for compactness of interpolated operators by the limit J-method.

▷ F. Cobos, L.M. Fernández-Cabrera, A. Martínez, to appear in "Spectral Theory, Function Spaces and Inequalities", Birkhäuser (to appear).

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(i) $\quad \beta(T_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,q;J}}) = 0 \quad$ if $\quad \beta(T_{A_0,B_0}) = 0,$

(ii) $\quad \beta(T_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,q;J}}) \le C\beta(T_{A_0,B_0}) \quad$ if $\quad \beta(T_{A_1,B_1}) = 0,$

(iii) $\quad \beta(T_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,q;J}}) \le C\beta(T_{A_0,B_0})\Big(1 + \Big(\log \frac{\beta(T_{A_1,B_1})}{\beta(T_{A_0,B_0})}\Big)_+\Big)$ if $\beta(T_{A_i,B_i}) > 0$ for $i = 0, 1$.

Here C is a constant independent of T.

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• Equivalence Theorem. *Description with the dual functional.*

THEOREM.- Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 < q \leq \infty$. Then

$$(A_0, A_1)_{0,q;J} = \Big\{ a \in A_1 : \|a\| = \Big(\int_1^\infty \Big[\frac{K(t,a)}{1+\log t} \Big]^q \frac{dt}{t} \Big)^{1/q} < \infty \Big\}.$$

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$\rightsquigarrow$ A logarithmic factor should be inserted.

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THEOREM.- Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \leq q < \infty$. Then $(A_0, A_1)_{1,q;K}$ is formed by all those $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that $a = \int_1^\infty u(t) \frac{dt}{t}$ (convergence in A_1) and

$$\Big(\int_1^\infty \Big(t^{-1} (1 + \log t) J(t, u(t)) \Big)^q \frac{dt}{t} \Big)^{1/q} < \infty \, .$$

Moreover

$$\|a\|_{1,q;K} \sim \inf \Big\{ \Big(\int_1^\infty \Big(t^{-1} (1 + \log t) J(t, u(t)) \Big)^q \frac{dt}{t} \Big)^{1/q} \Big\}$$

where the infimum is taken over all representations $u(t)$ as above.

Previous results on K-spaces defined using logarithmic terms:

- ▷ W.D. Evans and B. Opic, Canad. J. Math. 52 (2000) 920-960.
- ▷ W.D. Evans, B. Opic and L. Pick, J. of Inequal. Appl. 7 (2002) 187-269.
- ▷ A. Gogatishvili, B. Opic and W. Trebels, Math. Nachr. 278 (2005) 86-107.

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Notation

$L_0(t) = t\,,\ L_1(t) = \log t$ and $L_j(t) = \log(L_{j-1}(t))$ if $j > 1$

and

$E_0(t) = t\,,\ E_1(t) = e^t$ and $E_j(t) = e^{E_{j-1}(t)}$ if $j > 1.$

DEFINITION.- Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$. Suppose that $1 \leq q \leq \infty$, $m \in \mathbb{N} \cup \{0\}$ and let $\bar{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_m) \in \mathbb{R}^{m+1}$ such that

$$\left(\int_{E_m(1)}^{\infty} \Big(\frac{1}{\prod_{j=0}^{m} L_j(t)^{\alpha_j}} \Big)^q \frac{dt}{t} \right)^{1/q} < \infty \,. \tag{3}$$

The K-space $(A_0, A_1)_{\bar{\alpha},q;K}$ consists of all $a \in A_1$ which have a finite norm

$$\|a\|_{\bar{\alpha},q;K} = \left(\int_{E_m(1)}^{\infty} \Big(\frac{K(t,a)}{\prod_{j=0}^{m} L_j(t)^{\alpha_j}} \Big)^q \frac{dt}{t} \right)^{1/q} \,.$$

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- If (3) does not hold, then $(A_0, A_1)_{\bar{\alpha},q;K} = \{0\}$.

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- If $\alpha_0 > 1$ then $(A_0, A_1)_{\bar{\alpha},q;K} = A_1$.

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- The case $0 < \alpha_0 < 1$ corresponds to the real method with a function parameter studied in

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- Our interest is in the limiting cases where $\alpha_0 = 0$ and $\alpha_1 > 1/q$ or $\alpha_0 = 0\,, \alpha_1 = \cdots = \alpha_{r-1} = 1/q$ and $\alpha_r > 1/q$ for some $2 \leq r \leq m$. Condition (3) is satisfied in these cases.

THEOREM.- Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$, let $1 \le q \le \infty$, $m \in \mathbb{N} \cup \{0\}$ and let $\bar\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m) \in \mathbb{R}^{m+1}$ with $\alpha_0 = 0$ and $\alpha_1 > 1/q$, (in this case we let $r = 1$) or $\alpha_0 = 0$, $\alpha_1 = \cdots = \alpha_{r-1} = 1/q$ and $\alpha_r > 1/q$ for some $2 \le r \le m$. Put

$$\Psi(t) = \prod_{j=1}^{r} L_j(t)\,.$$

Then $(A_0, A_1)_{\bar\alpha,q;K}$ is formed by all those elements $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that

$$a = \int_{E_m(1)}^{\infty} u(t)\frac{dt}{t} \quad \text{(convergence in } A_1\text{)} \tag{4}$$

and

$$\Big(\int_{E_m(1)}^{\infty}\Big(\frac{\Psi(t)J(t,u(t))}{\prod_{j=0}^{m} L_j(t)^{\alpha_j}}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty \tag{5}$$

Moreover

$$\|a\|_{\bar\alpha,q;K} \sim \inf\left\{\Big(\int_{E_m(1)}^{\infty}\Big(\frac{\Psi(t)J(t,u(t))}{\prod_{j=0}^{m} L_j(t)^{\alpha_j}}\Big)^q\frac{dt}{t}\Big)^{1/q}\right\}$$

where the infimum is taken over all representations u satisfying (4) and (5).

THEOREM.- Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$, let $1 \leq q \leq \infty$ and $\alpha_1 > 1/q$. The K-space

$$(A_0, A_1)_{(0,\alpha_1),q;K} = \left\{ a \in A_1 : \|a\| = \left(\int_e^\infty \left[\frac{K(t,a)}{(\log t)^{\alpha_1}} \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

is formed by all those $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that

$$a = \int_e^\infty u(t) \frac{dt}{t}$$

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$$\left(\int_e^\infty \left(\frac{\log t J(t, u(t))}{(\log t)^{\alpha_1}} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \,.$$

THEOREM.- Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \leq q \leq \infty$ and $\alpha_2 > 1/q$. The K-space

$$(A_0, A_1)_{(0,1/q,\alpha_2),q;K} = \Big\{ a \in A_1 : \|a\| = \Big(\int_{e^e}^{\infty} \Big[\frac{K(t,a)}{(\log t)^{1/q}(\log(\log t))^{\alpha_2}} \Big]^q \frac{dt}{t} \Big)^{1/q} < \infty \Big\}$$

is formed by all those $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that

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and

$$\Big(\int_{e^e}^{\infty} \Big(\frac{(\log t)^{1/q'} J(t, u(t))}{(\log(\log t))^{\alpha_2 - 1}} \Big)^q \frac{dt}{t} \Big)^{1/q} < \infty \, .$$

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- Formulae for couples of vector-valued sequence spaces

▷ C. Geiss and S. Geiss, Stochastics and Stochastics Reports 76 (2004) 339-362.

▷ S. Geiss and M. Hujo, J. Approx. Theory 144 (2007) 213-232.

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