V. I. Burenkov

SHARP SPECTRAL STABILITY ESTIMATES FOR UNIFORMLY ELLIPTIC DIFFERENTIAL OPERATORS

Joint work with P.D. Lamberti

Introduction

We consider the eigenvalue problem for the operator

$$Hu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u \right), \quad x \in \Omega,$$

subject to homogeneous Dirichlet or Neumann boundary conditions, where $m \in \mathbb{N}$, Ω is a bounded open set in $\mathbb{R}^N$ and the coefficients $A_{\alpha\beta}$ are real-valued Lipschitz continuous functions satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \theta|\xi|^2$$

for all $x \in \Omega$ and for all $\xi_\alpha \in \mathbb{R}$, $|\alpha| = m$.

We consider open sets Ω for which the spectrum is discrete and can be represented by means of a non-decreasing sequence of non-negative eigenvalues of finite multiplicity

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \ldots$$

Here each eigenvalue is repeated as many times as its multiplicity and

$$\lim_{n\to\infty} \lambda_n[\Omega] = \infty\,.$$

The aim is sharp estimates for the variation

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$$

of the eigenvalues corresponding to two open sets Ω_1, Ω_2.

There is vast literature on spectral stability problems for elliptic operators. However, very little attention has been devoted to the problem of spectral stability for higher order operators and in particular to the problem of finding explicit qualified estimates for the variation of the eigenvalues. Moreover, most of the existing qualified estimates for second order operators were obtained under certain regularity assumptions on the boundaries.

Our analysis comprehends

operators of arbitrary even order,

with homogeneous Dirichlet or Neumann boundary conditions,

and

open sets admitting arbitrarily strong degeneration.

Preliminaries and notation

Let $N, m \in \mathbb{N}$ and Ω be an open set in $\mathbb{R}^N$. We denote by $W^{m,2}(\Omega)$ the Sobolev space of complex-valued functions in $L^2(\Omega)$, which have all distributional derivatives up to order m in $L^2(\Omega)$, endowed with the norm

$$\|u\|_{W^{m,2}(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^2(\Omega)}.$$

We denote by $W_0^{m,2}(\Omega)$ the closure in $W^{m,2}(\Omega)$ of the space of the C^∞-functions with compact support in Ω.

Let $V(\Omega)$ be a closed subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$. We consider the following eigenvalue problem

$$\begin{aligned} Q_\Omega(u,v) \equiv \int_\Omega \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta} D^\alpha u D^\beta \bar{v} dx \\ = \lambda \int_\Omega u\bar{v} dx, \end{aligned}$$

for all test functions $v \in V(\Omega)$, in the unknowns $u \in V(\Omega)$ (the eigenfunctions) and $\lambda \in \mathbb{R}$ (the eigenvalues). This is the weak formulation of the eigenvalue problem for the operator H subject to suitable homogeneous boundary conditions: the choice of $V(\Omega)$ corresponds to the choice of the boundary conditions.

Let $V(\Omega)$ be a closed subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$ and such that the embedding

$$V(\Omega) \subset W^{m-1,2}(\Omega)$$

is compact.

Then *there exists a non-negative self-adjoint linear operator* $H_{V(\Omega)}$ *in* $L^2(\Omega)$ *with compact resolvent, such that* $\mathrm{Dom}(H_{V(\Omega)}^{1/2}) = V(\Omega)$ *and*

$$\int\limits_{\Omega} H_{V(\Omega)}^{1/2} u \, \overline{H_{V(\Omega)}^{1/2} v} \, dx = Q_\Omega(u,v)$$

for all $u, v \in V(\Omega)$.

Moreover, the eigenvalues of the problem under consideration coincide with the eigenvalues $\lambda_n[H_{V(\Omega)}]$ of $H_{V(\Omega)}$ and

$$\lambda_n[H_{V(\Omega)}] = \inf_{\substack{\mathcal{L} \subset V(\Omega) \\ \dim \mathcal{L} = n}} \sup_{\substack{u \in \mathcal{L} \\ u \neq 0}} \frac{Q_\Omega(u,u)}{\|u\|_{L^2(\Omega)}^2},$$

where the infimum is taken with respect to all subspaces $\mathcal{L}$ of $V(\Omega)$ of dimension n (the Courant Min-Max principle).

If the embedding

$$W_0^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$$

is compact, we set

$$\lambda_{n,\mathcal{D}}[\Omega] = \lambda_n[H_{W_0^{m,2}(\Omega)}].$$

If the embedding

$$W^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$$

is compact, we set

$$\lambda_{n,\mathcal{N}}[\Omega] = \lambda_n[H_{W^{m,2}(\Omega)}].$$

The numbers $\lambda_{n,\mathcal{D}}[\Omega]$, $\lambda_{n,\mathcal{N}}[\Omega]$ are called the Dirichlet eigenvalues, Neumann eigenvalues respectively.

Example. If $Hu = -\Delta U$, then

$$\lambda_{n,\mathcal{D}} = \inf_{\substack{\mathcal{L} \subset W_0^{1,2} \\ \dim \mathcal{L} = n}} \sup_{\substack{u \in \mathcal{L} \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

and

$$\lambda_{n,\mathcal{N}} = \inf_{\substack{\mathcal{L} \subset W^{1,2} \\ \dim \mathcal{L} = n}} \sup_{\substack{u \in \mathcal{L} \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

Open sets with continuous boundaries

We consider bounded open sets whose boundaries are just locally the subgraphs of continuous functions.

For any set V in $\mathbb{R}^N$ and $\delta > 0$

$$V_\delta = \{x \in V : \ d(x, \partial\Omega) > \delta\},$$

where $d(x, \partial\Omega)$ is the distance of x to the boundary $\partial\Omega$.

Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \le s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$.

We say that that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an *atlas* in $\mathbb{R}^N$.

We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:

(i) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \ne \emptyset$;

(ii) $V_j \cap \partial\Omega \ne \emptyset$ for $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ for $s' < j \le s$;

(iii) for $j = 1, ..., s$

$$r_j(V_j) = \{\, x \in \mathbb{R}^N : \; a_{ij} < x_i < b_{ij},\; i = 1, ..., N\},$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \; a_{Nj} < x_N < g_j(\bar{x}),\; \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, ..., x_{N-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \; a_{ij} < x_i < b_{ij},\; i = 1, ..., N-1\}$ and g_j is a continuous function defined on $\overline{W}_j$ (it is meant that if $s' < j \leq s$ then $g_j(\bar{x}) = b_{Nj}$ for all $\bar{x} \in \overline{W}_j$);

moreover for $j = 1, \ldots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho,$$

for all $\bar{x} \in \overline{W}_j$.

For all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ we define the *atlas distance* $d_{\mathcal{A}}$ by

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) = \max_{j=1,\ldots,s} \; \sup_{(\bar{x}, x_N) \in r_j(V_j)} |g_{1j}(\bar{x}) - g_{2j}(\bar{x})| \; .$$

Given $k \in \mathbb{N}$ and $M > 0$ we denote by $C^k_M(\mathcal{A})$ the set of all $\Omega \in C(\mathcal{A})$ for which

$$\max_{j=1,\ldots,s'} \; \max_{1 \leq |\alpha| \leq k} \; \max_{\bar{x} \in \overline{W}_j} |D^\alpha g_j(\bar{x})| \leq M \, ,$$

and we denote by $C^{k,1}_M(\mathcal{A})$ the set of all $\Omega \in C(\mathcal{A})$ for which apart from this

$$|D^\alpha g_j(\bar{x}) - D^\alpha g_j(\bar{y})| \leq M |\bar{x} - \bar{y}|$$

for all $\bar{x}, \bar{y} \in \overline{W}_j$ and $j = 1, \ldots, s'$.

We shall always assume that an atlas $\mathcal{A}$ is fixed and all open sets Ω under consideration belong to $C(\mathcal{A})$.

Theorem 1.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$.

Let $m \in \mathbb{N}$, $L, \theta > 0$

For all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha| = |\beta| = m$, let $A_{\alpha\beta} \in C^{0,1}(\cup_{j=1}^s V_j)$ satisfy $A_{\alpha\beta} = A_{\beta\alpha}$,

$$\|A_{\alpha\beta}\|_{C^{0,1}(\cup_{j=1}^s V_j)} \leq L$$

and the ellipticity condition.

Then for each $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, \theta$ such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) < \varepsilon_n \, .$$

Estimates via the lower Hausdorff-Pompeiu deviation

If $C \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$ we denote by $d(x, C)$ the euclidean distance of x to C.

Let $A, B \subset \mathbb{R}^N$. We define the lower Hausdorff-Pompeiu deviation of A from B by

$$d_{\mathcal{HP}}(A, B) = \min\left\{\sup_{x\in A} d(x, B),\ \sup_{x\in B} d(x, A)\right\}.$$

If the minimum is replaced by the maximum, then the right-hand side becomes the usual Hausdorff-Pompeiu distance $d^{\mathcal{HP}}(A, B)$ of A and B.

We now introduce a class of open sets for which we can estimate the atlas distance $d_{\mathcal{A}}$ via the lower Hausdorff-Pompeiu deviation of the boundaries.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $\omega : [0, \infty[\to [0, \infty[$ be a continuous non-decreasing function such that $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \geq kt$ for all $0 \leq t \leq 1$.

Let $M > 0$. We denote by $C_M^{\omega(\cdot)}(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ belonging to $C(\mathcal{A})$ and such that all the functions g_j in the part (iii) of the definition of an open set of class $C(\mathcal{A})$ satisfy the condition

$$|g_j(\bar{x}) - g_j(\bar{y})| \leq M\omega(|\bar{x} - \bar{y}|),$$

for all

$$\bar{x}, \bar{y} \in \overline{W}_j\,.$$

Theorem 2.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let the assumptions of Theorem 1 on the coefficients $A_{\alpha\beta}$ be satisfied.

Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous non-decreasing function satisfying $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \geq kt$ for all $0 \leq t \leq 1$.

Then for each $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, M, \theta, \omega$ such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)),$$

for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying

$$d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2) < \varepsilon_n \, .$$

Corollary

Under the assumptions of Theorem 2 for each $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, M, \theta, \omega$ such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \le c_n \omega(\varepsilon),$$

for all $0 < \varepsilon < \varepsilon_n$ and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying the inclusion

$$(\Omega_1)_\varepsilon \subset \Omega_2 \subset (\Omega_1)^\varepsilon$$

or the inclusion

$$(\Omega_2)_\varepsilon \subset \Omega_1 \subset (\Omega_2)^\varepsilon .$$

Estimates via the measure of the symmetric difference

Theorem 3.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let the assumptions of Theorem 1 on the coefficients $A_{\alpha\beta}$ be satisfied.

Let $2 < p \leq \infty$ and let $\mathfrak{A}$ be a family of open sets of class $C_M^{m-1,1}(\mathcal{A})$ such that for each $n \in \mathbb{N}$

$$\sup_{\Omega \in \mathfrak{A}} \|\varphi_n[\Omega]\|_{W^{m,p}(\Omega)} < \infty,$$

for all $n \in \mathbb{N}$

Then for each $n \in \mathbb{N}$ there exists $c_n, \varepsilon_n > 0$ depending only on $n, \mathcal{A}, m, M, \theta$, p,

$$\sup_{\Omega \in \mathfrak{A}} \|\varphi_k[\Omega]\|_{W^{m,p}(\Omega)}\,, k = 1, \dots, n\,,$$

such that for the Dirichlet boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n |\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}},$$

where $|\Omega_1 \vartriangle \Omega_2|$ is the Lebesgue measure of the symmetric difference of Ω_1 and Ω_2, for all $\Omega_1, \Omega_2 \in \mathfrak{A}$ such that $|\Omega_1 \vartriangle \Omega_2| < \varepsilon_n$.

Moreover, the exponent $1 - \frac{2}{p}$ is sharp. It cannot be replaced by $1 - \frac{2}{p} + \delta$ where $\delta > 0$ is a constant independent of p.

Corollary.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let the assumptions of Theorem 1 on the coefficients $A_{\alpha\beta}$ be satisfied.

Then for all $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, \mathcal{A}, m, L, M, \theta$ such that for the Dirichlet boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n |\Omega_1 \vartriangle \Omega_2|,$$

for all $\Omega_1, \Omega_2 \in C_M^{2m}(\mathcal{A})$ satisfying

$$|\Omega_1 \vartriangle \Omega_2| < \varepsilon_n \, .$$

Dependence of c_n on n

Theorem 4.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$, $m = 1$ and let the assumptions of Theorem 1 on the coefficients $A_{\alpha\beta}$ with $m = 1$ be satisfied.

Then there exist $c, E > 0$ depending only on $N, \mathcal{A}, L, \theta$ such that for the Dirichlet boundary conditions for each $n \in \mathbb{N}$

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c\lambda_n[\Omega_1 \cap \Omega_2]\, d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) < E\,.$$

Estimates for the p-Laplacian

Let Ω be an open set in $\mathbb{R}^N$ of finite measure and $1 < p < \infty$. Consider the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u$$

for $u \in W_0^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$, where

$$\Delta_p u = \mathrm{div} |\nabla u|^{p-2} \nabla u$$

is the p-Laplacian. Clearly Δ_2 is the usual Dirichlet Laplacian. The real numbers λ for which this equation has a nontrivial solution are by definition the eigenvalues of $-\Delta_p$.

As is known, it is possible to produce a nondecreasing unbounded sequence of eigenvalues $\lambda_{p,n}[\Omega]$, $n \in \mathbb{N}$ by means of the Lusternik-Shnirelman Min-Max principle. Namely,

$$\lambda_{p,n}[\Omega] = \inf_{\mathcal{M} \in \mathfrak{M}_{p,n}(\Omega)} \sup_{u \in \mathcal{M}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

where $\mathfrak{M}_{p,n}(\Omega)$ is the family of those conic subsets $\mathcal{M}$ of $W_0^{1,p}(\Omega) \setminus \{0\}$, whose intersection with the unit sphere of $L^p(\Omega)$ is compact in $W_0^{1,p}(\Omega)$ and whose Krasnoselskii's genus $\gamma(\mathcal{M})$ is greater than or equal to n.

Theorem 5.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let and $1 < p < \infty$.

Then there exist $c, E > 0$ depending only on $N, \mathcal{A}$ and p such that for the Dirichlet boundary conditions for each $n \in \mathbb{N}$

$$|\lambda_{p,n}[\Omega_1] - \lambda_{p,n}[\Omega_2]| \leq c\lambda_{p,n}[\Omega_1 \cap \Omega_2]\, d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying

$$d_{\mathcal{A}}(\Omega_1, \Omega_2) < E\,.$$

Ideas of proofs

All proofs are based on the Courant Min-Max principle or, in the case of the p-Laplacian, on the Lusternik-Shnirelman Min-Max principle. P.D. Lamberti and I worked out the *method of transition operators*. From the point of view of this method in order to prove that or other spectral stability estimate for the eigenvalues it is required to construct an appropriate transition operator T_{12} from operator $H(\Omega_1)$ which is the initial operator H considered on the open set Ω_1 to the operator $H(\Omega_2)$ which is the operator H considered on the open set Ω_2.

"Ideal" transition operator is the one that transforms the eigenfunctions $\varphi_n[H(\Omega_1)]$ into the eigenfunctions $\varphi_n[H(\Omega_2)]$, $n \in \mathbb{N}$. However, this requires to know all eigenfunctions $\varphi_n[H(\Omega_1)]$ and $\varphi_n[H(\Omega_2)]$ which happens only in trivial cases. (Moreover, in such cases all eigenvalues $\lambda_n[H_1]$ and $\lambda_n[H_2]$ are also known and there is no problem in estimating $|\lambda_n[H(\Omega_1)] - \lambda_n[H(\Omega_2)]|$.) Hence one should try to use such transition operators which are in some sense close to the "ideal" transition operator if Ω_1 and Ω_2 are close, and can be constructed without knowing the eigenfunctions.

References

[1] V.I. Burenkov, E.B. Davies, Spectral stability of the Neumann Laplacian, *J. Differential Equations*, **186**, pp. 485-508, 2002.

[2] V.I. Burenkov, P.D. Lamberti, Spectral stability of general non-negative self-adjoint operators with applications to Neumann-type operators, *J. Differential Equations*, **233**, 345-379, 2007.

[3] V.I. Burenkov, P.D. Lamberti, Spectral stability of Dirichlet second order uniformly elliptic operators, *J. Differential Equations*, **244**, pp. 1712-1740, 2008.

[4] Burenkov V. I., Lamberti P. D., Spectral stability of higher order uniformly elliptic differential operators. Sobolev spaces in mathematics. II, 69-102, Int. Math. Ser. (N. Y.), **9** (2009), Springer, New York.

[5] Burenkov V. I., Lamberti P. D., Spectral stability of the p-Laplacian. Nonlinear Anal. **71** (2009), no. 5-6, 2227-2235.

[6] Burenkov V. I., Lamberti P. D., Spectral stability estimates via the Lebesgue measure of domains for higher order elliptic operators. Revista Matemática Compultese (accepted for publication).