Recent progresses on superposition in Besov-Lizorkin-Triebel spaces

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1 Introduction

For a given real valued function space E, the Superposition Operator Problem (S.O.P.) consists in the characterization of all the functions $f : \mathbb{R} \to \mathbb{R}$ such that the superposition operator

$$T_f : \ g \mapsto f \circ g \,,$$

takes E to itself.
This paper is devoted to the S.O.P. in Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$ and Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, denoted simply by $E^s_{p,q}(\mathbb{R}^n)$ in case there is no need for a distinction. In the same spirit, $E^s_{p,q}(\mathbb{R}^n)$ can denote $W^s_p(\mathbb{R}^n)$ in case $s \in \mathbb{N}$ and $p = 1$ or $+\infty$, despite those spaces are not Besov nor Lizorkin-Triebel spaces. In all the paper we assume that s is a positive real number, and that $p, q \in [1, \infty]$ ($p < \infty$ in L-T case). We consider the following problems:

Full S.O.P.
Describe the class of all functions f having the acting property

$$T_f(E^s_{p,q}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)\,. \tag{1}$$

Restricted S.O.P. :
Describe the class of all functions f having the restricted acting property

$$T_f(E^s_{p,q} \cap L_\infty(\mathbb{R}^n)) \subset E^s_{p,q} \cap L_\infty(\mathbb{R}^n)\,. \tag{2}$$

The difference between both problems was first observed by Dahlberg [15]: *if s is an integer such that $1+(1/p) < s < n/p$, then only linear functions have the acting property on $W^s_p(\mathbb{R}^n)$.* This degeneracy property was extended to all spaces $E^s_{p,q}(\mathbb{R}^n)$ by the author [2, 4] and by Thomas Runst [19]. On the contrary, $E^s_{p,q} \cap L_\infty(\mathbb{R}^n)$ is known to be a Banach algebra for the pointwise product. Hence any entire function possesses the restricted acting property. For a more complete overview, we refer to the survey papers [9, 14].

2 Necessary conditions

All were found essentially before 1995. They give rise to conjectures concerning both problems.

Proposition 1 *The property $T_f(\mathcal{D}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)$ implies $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$.*

Proposition 2 (i) *The property $T_f(E^s_{p,q} \cap L_\infty(\mathbb{R}^n)) \subset B^s_{p,\infty}(\mathbb{R}^n)$ implies $f \in W^{1,\ell oc}_\infty(\mathbb{R})$ (i.e. f locally Lipschitz continuous).*
(ii) *In case $E^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, the property $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset B^s_{p,\infty}(\mathbb{R}^n)$ implies that f is uniformly Lipschitz continuous.*

For a proof of Proposition 2, see [6].

Remarks:
1- The embedding $W^{1,\ell oc}_\infty(\mathbb{R}) \hookrightarrow E^{s,\ell oc}_{p,q}(\mathbb{R})$ holds if $0 < s < 1$, while the reverse embedding holds for $s > 1+(1/p)$. Hence the values $s = 1$, $s = 1+(1/p)$ and $s = n/p$ appear as the critical ones for superposition problem.
2- The superposition problem has been solved before 1990 in case $0 < s < 1$: the Lipschitz conditions of Proposition 2 are sufficient.

3 Case $s > 1 + (1/p)$, restricted problem

Conjecture 1 *If $s > 1+(1/p)$, then T_f maps $E^s_{p,q} \cap L_\infty(\mathbb{R}^n)$ to $E^s_{p,q}(\mathbb{R}^n)$ iff $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$ and $f(0) = 0$.*

The conjecture has been proved:

- around 1990 for the usual Sobolev spaces $W^s_p(\mathbb{R}^n)$, s integer (talk at FSDONA 1992, see [3, 5]),
- between 2005 and 2010 for fractional spaces if $n = 1$ and $p > 1$ (also for $F^s_{1,q}(\mathbb{R})$, if s is not an integer). See [7, 11, 12, 13, 17].

3.1 Case $s > 1 + (1/p)$, $n = 1$

Theorem 1 *If $s > 1 + (1/p)$ and $p > 1$, then T_f maps $E^s_{p,q}(\mathbb{R})$ to itself iff $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$ and $f(0) = 0$.*

We give a sketchy proof in the simplest case:

Proposition 3 *If $1 < p \leq q$ and $1 + (1/p) < s < 2$, there exists a constant $c > 0$ such that*

$$\| f \circ g \|_{B^s_{p,q}} \leq c \| f' \|_{B^{s-1}_{p,q}} \left(\| g \|_{B^s_{p,q}} + \| g' \|^{s-(1/p)}_{BV_{sp-1}} \right) \tag{3}$$

holds for all f such that $f' \in B^{s-1}_{p,q}(\mathbb{R})$ and $f(0) = 0$, and all $g \in B^s_{p,q}(\mathbb{R})$.

3.1.1 A preparation

We need to introduce the space BV_p, and various equivalent norms in the Besov space.
A function $g : \mathbb{R} \to \mathbb{R}$ is said to be of *bounded p-variation* if

$$\|g\|_{BV_p} := \sup \left(\sum_{k=1}^{N} |g(b_k) - g(a_k)|^p \right)^{1/p} < +\infty \,,$$

where the supremum runs over all finite sets $\{]a_k, b_k[\,;\, k = 1, \ldots, N\}$ of pairwise disjoint intervals. The set of all such functions is denoted by BV_p. It was first considered by Wiener [20]. It is related to Besov spaces by the Peetre's embedding [18]:

$$B^{1/p}_{p,1}(\mathbb{R}) \hookrightarrow BV_p(\mathbb{R}) \,.$$

As a consequence $B^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow BV_{sp-1}(\mathbb{R})$ holds for $s > 1 + (1/p)$.
We introduce the following notation:

$$\Delta_h f(x) := f(x + h) - f(x) \,,$$

$$\omega_p(f,h) := \left(\int_{\mathbb{R}} |\Delta_h f(x)|^p \, dx \right)^{1/p} \quad , \quad \forall h \in \mathbb{R}\,,$$

$$\Omega_p(f,t) := \left(\int_{\mathbb{R}} \sup_{|h|\leq t} |\Delta_h f(x)|^p \, dx \right)^{1/p} \quad , \quad \forall t > 0\,.$$

The following are equivalent norms in $B^s_{p,q}(\mathbb{R})$:

$$\| \, f \, \|_p + \left(\int_{|h|<R} \left(\frac{\omega_p(f,h)}{|h|^s} \right)^q \frac{dh}{|h|} \right)^{1/q} \,,$$

in case $0 < s < 1$, for any fixed $R \in]0, +\infty]$;

$$\| \, f \, \|_p + \left(\int_0^{\infty} \left(\frac{\Omega_p(f,t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} \,,$$

in case $1/p < s < 1$.

3.1.2 Case $1 + (1/p) < s < 2$, $n = 1$. Proof of Proposition 3

We are reduced to estimate

$$U(h) := \left(\int_{\mathbb{R}} |\Delta_h (f' \circ g)(x)|^p |g'(x)|^p \, dx \right)^{1/p} \,.$$

Without major loss of generality, we assume that $h > 0$ and that g is a real analytic function in $B^s_{p,q}(\mathbb{R})$. The complement in $\mathbb{R}$ of the set of zeros of g' is the union of a family $\{I_\ell\}$ of open disjoint intervals. We denote by $I'_\ell(h)$ the set of $x \in I_\ell$ whom distance to the right endpoint of I_ℓ is greater than h, and we set

$$I''_\ell(h) := I_\ell \setminus I'_\ell(h) \quad , \quad a_\ell := \sup_{I_\ell} |g'| \,,$$

$$U_1(h) := \left(\sum_\ell \int_{I'_\ell(h)} |\Delta_h (f' \circ g)(x)|^p |g'(x)|^p \, dx \right)^{1/p} \,,$$

and $U_2(h)$ similarly, with $I'_\ell(h)$ replaced by $I''_\ell(h)$.

Estimation of $U_1(h)$.

By the change of variable $y = g(x)$ on $I'_\ell(h)$, it holds

$$\int_{I'_\ell(h)} |\Delta_h(f' \circ g)(x)|^p \, |g'(x)|^p \, dx \le a_\ell^{p-1} \Omega_p^p(f', a_\ell h) \,.$$

By Minkowski inequality (using $q/p \ge 1$), we obtain

$$\left(\int_0^\infty \left(\frac{U_1(h)}{h^{s-1}} \right)^q \frac{dh}{h} \right)^{1/q} \le c \, \|f'\|_{B^{s-1}_{p,q}} \left(\sum_\ell |a_\ell|^{sp-1} \right)^{1/p} \,.$$

Since g' vanishes at the endpoints of I_ℓ, it holds

$$\left(\sum_\ell |a_\ell|^{sp-1} \right)^{1/p} \le \|g'\|^{s-(1/p)}_{BV_{sp-1}} \,.$$

Estimation of $U_2(h)$.
Since g' vanishes at the right endpoint of $I''_\ell(h)$, it holds

$$|g'(x)| \le \sup_{|v| \le h} |\Delta_v g'(x)|$$

for all $x \in I''_\ell(h)$. Thus we obtain

$$\sum_\ell \int_{I''_\ell(h)} |\Delta_h(f' \circ g)(x)|^p |g'(x)|^p \, dx \le (2\|f'\|_\infty)^p \, \Omega_p(g', h)^p \,.$$

Hence

$$\left(\int_0^\infty \left(\frac{U_2(h)}{h^{s-1}} \right)^q \frac{dh}{h} \right)^{1/q} \le c \, \|f'\|_\infty \, \|g'\|_{B^{s-1}_{p,q}} \,.$$

3.2 Case $s > 1 + (1/p)$, $n > 1$, restricted problem

The following partial result can be viewed as a good approximation of Conjecture 1 in the n-dimensional case:

Theorem 2 *Assume $s > 1$ and $1/p < s - [s]$. If $f(0) = 0$ and $f \in B^{s+\epsilon,\ell oc}_{p,\infty}(\mathbb{R}^n)$, for some $\epsilon > 0$, then*

$$T_f \left(B^s_{p,q} \cap L_\infty(\mathbb{R}^n) \right) \subset B^s_{p,q} \cap L_\infty(\mathbb{R}^n) \,.$$

For a proof, see [8].

4 Case $s > 1 + (1/p)$, full problem

If $s > \max(n/p, 1 + (1/p))$, the full problem reduces to the restricted one, by Sobolev embedding $E^s_{p,q}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$. If $1+(1/p) < s < n/p$, we are in the "triviality area" of Dahlberg. In the remaining case, we have the following:

Conjecture 2 *If $s = n/p > 1 + (1/p)$ and $E^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, then T_f maps $E^s_{p,q}(\mathbb{R}^n)$ to itself iff $f(0) = 0$ and f' belongs to $E^{s-1}_{p,q}(\mathbb{R})$ locally uniformly.*

Conjecture 2 was proved in 1990 for the usual Sobolev spaces [5]. In the general case:

- Necessity was proved recently by Allaoui and the author [1].
- Sufficiency is completely open.

5 Case $1 < s < 1 + (1/p)$

Conjecture 3 *Let $1 < s < 1 + (1/p)$.*

- *T_f maps $E^s_{p,q} \cap L_\infty(\mathbb{R}^n)$ to $E^s_{p,q}(\mathbb{R}^n)$ iff $f \in E^{s,\ell oc}_{p,q} \cap W^{1,\ell oc}_\infty(\mathbb{R})$ and $f(0) = 0$.*
- *In case $E^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, T_f maps $E^s_{p,q}(\mathbb{R}^n)$ to itself iff $f(0) = 0$, $f' \in L_\infty(\mathbb{R})$ and f' belongs to $E^{s-1}_{p,q}(\mathbb{R})$ locally uniformly.*

Very few is known concerning this conjecture. The best sufficient condition was obtained by Kateb [16]:

Theorem 3 *If $f' \in L_\infty \cap \dot{B}^{1/p}_{p,\infty}(\mathbb{R})$ and $f(0) = 0$, then T_f maps $B^s_{p,q}(\mathbb{R}^n)$ to itself for all $0 < s < 1 + (1/p)$.*

Here we denote by $\dot{B}^s_{p,q}$ the *homogeneous* Besov space.

6 Remark on the critical values

In case $s = 1$ or $s = 1 + (1/p)$, more complicated acting conditions are expected, see the following result of Lanza and the author [10]:

Theorem 4 *T_f takes $B^1_{\infty,\infty}(\mathbb{R}^n)$ to itself iff f is locally Lipschitz continuous and*

$$f(x+t) + f(x-t) - 2f(x) = O\left(\frac{t}{|\log t|}\right), \qquad (4)$$

as $t \to 0+$, uniformly on each compact subset of $\mathbb{R}^n$.

Notice that condition (4) is stronger than $f \in B^{1,\ell oc}_{\infty,\infty}(\mathbb{R})$.

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