ON A NONHOMOGENEOUS ELLIPTIC EQUATION WITH DECAYING CYLINDRICAL POTENTIAL AND CRITICAL EXPONENT

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ABSTRACT. In this talk, we prove the existence and multiplicity of solutions for a nonhomogeneous elliptic equation involving decaying cylindrical potential and critical exponent.

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1. INTRODUCTION

In this talk, we consider the following problem $(\mathcal{P})$

$$\begin{cases} -\operatorname{div}\left(|y|^{-2a}\nabla u\right)-\mu\,|y|^{-2(a+1)}u= & h\,|y|^{-2_*b}\,|u|^{2_*-2}u+\lambda g(x) \text{ in } \mathbb{R}^N,\\ & y\neq 0\\ u\in\mathcal{D}_0^{1,2}, & \end{cases}$$

we denote a point x in $\mathbb{R}^N$ by the pair $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, k and N are integers such that $N\geq 3$ and k belongs to $\{2,...,N\}$, $-\infty<a<(k-2)/2$, $a\leq b<a+1$, $2_*=2N/(N-2+2(b-a))$, $-\infty<\mu<\bar{\mu}_{a,k}:=((k-2(a+1))/2)^2$, $g\in\mathcal{H}_\mu^{'}\cap C\left(\mathbb{R}^N\right)$, h is a bounded positive function on $\mathbb{R}^k$ and λ is a real parameter. $\mathcal{H}_\mu^{'}$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and $\mathcal{D}_0^{1,2}$ will be defined later.

It is clear that the degeneracy and singularity occur in the problem $(\mathcal{P})$. In such situations, standard variational methods are not applied.

The problem is related to the following inequality [7] which states that: there exists a positive constant $C_{a,b}$ such that

(1.1)
$$\left(\int_{\mathbb{R}^N}|y|^{-2_*b}\,|v|^{2_*}\,dx\right)^{2/2_*}\leq C_{a,b}\int_{\mathbb{R}^N}|y|^{-2a}\,|\nabla v|^2\,dx,\forall v\in C_c^\infty(\left(\mathbb{R}^k\setminus\{0\}\right)\times\mathbb{R}^{N-k})$$

where $-\infty<a<(N-2)/2$ and $a\leq b\leq a+1$.

If $b=a+1$ in (1.1), then $2_*=2$ and we have the following Hardy inequality with cylindrical weights (see [6])

$$\bar{\mu}_{a,k}\int_{\mathbb{R}^N}|y|^{-2(a+1)}\,v^2dx\leq\int_{\mathbb{R}^N}|y|^{-2a}\,|\nabla v|^2\,dx,\text{ for all } v\in\mathcal{H}_\mu, \tag{1.2}$$

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We denote by $\mathcal{D}_0^{1,2}=\mathcal{D}_0^{1,2}\left(\mathbb{R}^k\backslash\{0\}\times\mathbb{R}^{N-k}\right)$ and $\mathcal{H}_\mu=\mathcal{H}_\mu\left(\mathbb{R}^k\backslash\{0\}\times\mathbb{R}^{N-k}\right)$, the closure of $C_c^\infty(\left(\mathbb{R}^k\backslash\{0\}\right)\times\mathbb{R}^{N-k})$ with respect to the norms

$$\|u\|_{a,0}=\left(\int_{\mathbb{R}^N}|y|^{-2a}|\nabla u|^2\,dx\right)^{1/2}$$

and

$$\|u\|_{a,\mu}=\left(\int_{\mathbb{R}^N}\left(|y|^{-2a}|\nabla u|^2-\mu\,|y|^{-2(a+1)}|u|^2\right)dx\right)^{1/2},$$

with $\mu<\bar{\mu}_{a,k}$ respectively.

Using the above Hardy inequality, the norm $\|u\|_{a,\mu}$ is equivalent to $\|u\|_{a,0}$. More explicitly, we have, $\mu<\bar{\mu}_{a,k}$,

$$\left(1-\mu^+/\bar{\mu}_{a,k}\right)^{1/2}\|u\|_{a,0}\leq\|u\|_{a,\mu}\leq\left(1-\mu^-/\bar{\mu}_{a,k}\right)^{1/2}\|u\|_{a,0}\ \text{ for } u\in\mathcal{H}_\mu,$$

where $\mu^+=\max(\mu,0)$ and $\mu^-=\min(\mu,0)$.

Since our approach is variational, we define the energy functional on $\mathcal{H}_\mu$ by

$$I(u):=(1/2)\|u\|_{a,\mu}^2-(1/2_*)\int_{\mathbb{R}^N}h\,|y|^{-2_*b}|u|^{2_*}\,dx-\lambda\int_{\mathbb{R}^N}gudx.$$

We say that $u\in\mathcal{H}_\mu$ is a weak solution of the problem $(\mathcal{P})$ if it satisfies

$$\begin{aligned}\left\langle I'(u),v\right\rangle &= \int_{\mathbb{R}^N}\left(|y|^{-2a}\nabla u\nabla v-\mu\,|y|^{-2(a+1)}uv-h\,|y|^{-2_*b}|u|^{2_*-2}uv-\lambda gv\right)dx\\ &= 0,\ \text{for } v\in\mathcal{H}_\mu.\end{aligned}$$

Here $\langle .,.\rangle$ denotes the product in the duality $\mathcal{H}_\mu',\mathcal{H}_\mu$.

We give a brief historic:

In the spherical case i.e. $k=N$, the problem $(\mathcal{P})$ with $a=0, 0\leq\mu<\bar{\mu}_{0,N}$ and $h=1$, has been studied by Wang and Zhou [9]. By applying Ekeland's variational principle and mountain pass Theorem, they proved the existence of two distinct solutions under some conditions on g.

Using the Bliss Lemma, Xuan et al. [10] obtained also for $k=N$, an explicit form of the extremal functions associated to the best constant of the embedding $\mathcal{H}_\mu\hookrightarrow L^{2_*}(\mathbb{R}^N,|x|^{-2_*b}\,dx)$. They proved that if $0<\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}+a<(N-2)/2$, and $\mu<\overline{\mu}-b^2$ then for $\varepsilon>0$, the functions defined as

$$u_\varepsilon(x)=C_0\varepsilon^{\frac{2}{2_*-2}}\left(\varepsilon^{\frac{2\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}-\mu}-b}}|x|^{\frac{2_*-2}{2}\left(\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}\right)}+|x|^{\frac{2_*-2}{2}\left(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}\right)}\right)^{-\frac{2}{2_*-2}} \tag{1.3}$$

with a suitable positive constant C_0, are weak solutions of

$$-\text{div}\left(|x|^{-2a}\nabla u\right)-\mu\,|x|^{-2(a+1)}u=|x|^{-2_*b}|u|^{2_*-2}u\qquad\text{in }\ \mathbb{R}^N\backslash\{0\}.$$

Furthermore

$$\|u_\varepsilon\|_{\mu,a}^2=|u_\varepsilon|_{2_*,2_*b}^2=S_{\mu,N}^{N/(2(a+1-b))}, \tag{1.4}$$

where $S_{\mu,N}$ is the best constant defined as

$$S_{\mu,N}:=\inf_{u\in H_\mu\backslash\{0\}}\frac{\int_{\mathbb{R}^N}\left(|x|^{-2a}|\nabla u|^2-\mu\,|x|^{-2(a+1)}u^2\right)dx}{\left(\int_{\mathbb{R}^N}|x|^{-2_*b}|u|^{2_*}\,dx\right)^{2/2_*}}. \tag{1.5}$$

Here $\overline{\mu}$ and $|u|_{2_*,2_*b}$ denote $\bar{\mu}_{a,N}$ and the norm of $L^{2_*}(\mathbb{R}^N, |x|^{-2_*b}\,dx)$ respectively.

Bouchekif and Matallah [3] established, in the spherical case, the existence of multiple solutions of the problem $(\mathcal{P})$.

For the cylindrical case i.e., $k < N$, there are much less studies in the literature to our knowledge. We cite for example [1, 2, 5, 7, 8] and the references therein. Secchi et al. in [8] considered the minimization problem

$$\begin{aligned} S(p) &= S(N,p,k,s) = \\ &\inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p, u \in C_0^\infty\left(\left(\mathbb{R}^k \setminus \{0\}\right)\times \mathbb{R}^{N-k}\right) \text{ and } \int_{\mathbb{R}^N} |y|^{-s}|u|^q\,dx = 1\right\} \end{aligned}$$

where $0 \le s < k, s \le k, q = q(N,p,s) = \frac{p(N-s)}{N-p}$ and $1 < p < \infty$. They computed the optimal constant for a generalized Hardy-Sobolev inequality.

When $0 \le s < 2$, then $S(2)$ is achieved (see [2]). They proved that the optimal function for $S(2)$ satisfies $u = u^*(.,z)$ (the Schwarz symmetrization of $u(.,z)$).

Most previous studies based on the knowledge of the explicit form of the extremal functions of Hardy-Sobolev constant. Those of $S_{\mu,k}, k < N$, exist but are not known explicitly. For this, we consider the minimization of the energy functional on the natural constraint that is the variety of Nehari.

Throughout this work, we consider the following assumptions:

(G) There exist $\nu_0 > 0$ and $\delta_0 > 0$ such that $g(x) \ge \nu_0$, for all x in $B(0, 2\delta_0)$.

(H) $\lim\limits_{|y|\longrightarrow 0} h(y) = \lim\limits_{|y|\longrightarrow\infty} h(y) = h_0 > 0$, $h(y) \ge h_0$, $y \in \mathbb{R}^k$.

Here, $B(a,r)$ denotes the ball centered at a with radius r.

Under some sufficient conditions on coefficients of equation of $(\mathcal{P})$, we split $\mathcal{N}$ in two disjoint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$, thus we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$.

We shall state our main results:

Theorem 1. *Assume that $2 \le k \le N$, $-1 < a < (k-2)/2$, $\mu < \bar{\mu}_{a,k}$, and (G) holds, then there exists $\Lambda_1 > 0$ such that the problem $(\mathcal{P})$ has at least one nontrivial solution on $\mathcal{H}_\mu$ for all $\lambda \in (0, \Lambda_1)$.*

Theorem 2. *In addition to the assumptions of the Theorem 1, if (H) holds, then there exists $\Lambda_2 > 0$ such that the problem $(\mathcal{P})$ has at least two nontrivial solutions on $\mathcal{H}_\mu$ for all $\lambda \in (0, \Lambda_2)$.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2.

2. Preliminaries

Proposition 1. *(see [7]) $S_{\mu,k}$ defined by*

$$S_{\mu,k} := \inf_{v\in\mathcal{H}_\mu\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(|y|^{-2a}|\nabla v|^2 - \mu|y|^{-2(a+1)}v^2\right)dx}{\left(\int_{\mathbb{R}^N}|y|^{-2_*b}|v|^{2_*}\,dx\right)^{2/2_*}}, \tag{2.1}$$

is achieved on $\mathcal{H}_\mu$, for $2 \le k < N$ and $\mu < \bar{\mu}_{a,k}$.

Definition 1. *Let $c \in \mathbb{R}$, E a Banach space and $I \in C^1(E, \mathbb{R})$.*

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I(u_n) = c + o_n(1) \text{ and } I^{'}(u_n) = o_n(1),$$

where $o_n(1) \longrightarrow 0$ as $n \longrightarrow \infty$.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

2.1. **Nehari manifold.** It is well known that I is of class C^1 in $\mathcal{H}_\mu$ and the solutions of $(\mathcal{P})$ are the critical points of I which is not bounded below on $\mathcal{H}_\mu$. Consider the following Nehari manifold

$$\mathcal{N} = \left\{ u \in \mathcal{H}_\mu \backslash \{0\} : \left\langle I^{'}(u), u \right\rangle = 0 \right\},$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|_{a,\mu}^2 - \int_{\mathbb{R}^N} h |y|^{-2_* b} |u|^{2_*} dx - \lambda \int_{\mathbb{R}^N} g u dx = 0. \tag{2.2}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem $(\mathcal{P})$. Moreover, we have the following results.

Lemma 1. *I is coercive and bounded from below on $\mathcal{N}$.*

Define

$$\Psi_\lambda(u) = \left\langle I^{'}(u), u \right\rangle.$$

Then, for $u \in \mathcal{N}$

$$\begin{aligned} \left\langle \Psi_\lambda^{'}(u), u \right\rangle &= 2\|u\|_{a,\mu}^2 - 2_* \int_{\mathbb{R}^N} h |y|^{-2_* b} |u|^{2_*} dx - \lambda \int_{\mathbb{R}^N} g u dx \\ &= \|u\|_{a,\mu}^2 - (2_* - 1) \int_{\mathbb{R}^N} h |y|^{-2_* b} |u|^{2_*} dx \\ &= \lambda (2_* - 1) \int_{\mathbb{R}^N} g u dx - (2_* - 2) \|u\|_{a,\mu}^2 . \end{aligned} \tag{2.3}$$

Now, we split $\mathcal{N}$ in three parts:

$$\mathcal{N}^+ = \left\{ u \in \mathcal{N} : \left\langle \Psi_\lambda^{'}(u), u \right\rangle > 0 \right\}, \ \mathcal{N}^0 = \left\{ u \in \mathcal{N} : \left\langle \Psi_\lambda^{'}(u), u \right\rangle = 0 \right\}$$

and $\mathcal{N}^- = \left\{ u \in \mathcal{N} : \left\langle \Psi_\lambda^{'}(u), u \right\rangle < 0 \right\}$.

We have the following results.

Lemma 2. *Suppose that there exists a local minimizer u_0 for I on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then, $I^{'}(u_0) = 0$ in $\mathcal{H}_\mu^{'}$.*

Set

$$\Lambda_1 := (2_* - 2)(2_* - 1)^{-(2_* - 1)/(2_* - 2)} \left[(h_0)^{-1} S_{\mu,k} \right]^{2_*/2(2_* - 2)} \|g\|_{\mathcal{H}_\mu^{'}}^{-1} . \tag{2.4}$$

Lemma 3. *We have $\mathcal{N}^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.*

Define

$$c := \inf_{u \in \mathcal{N}} I(u), \ c^+ := \inf_{u \in \mathcal{N}^+} I(u) \text{ and } c^- := \inf_{u \in \mathcal{N}^-} I(u).$$

For the sequel, we need the following lemmas.

Lemma 4. *(i) If $\lambda \in (0, \Lambda_1)$, then one has $c \leq c^+ < 0$.*
(ii) If $\lambda \in (0, (1/2)\Lambda_1)$, then $c^- > C_1$, where

$$\begin{aligned} C_1 &= C_1\left(\lambda, S_{\mu,k}\|g\|_{\mathcal{H}'_\mu}\right) = ((2_* - 2)/2_* 2)(2_* - 1)^{2/(2_*-2)}(S_{\mu,k})^{2_*/(2_*-2)} + \\ &\qquad - \lambda(1 - (1/2_*))(2_* - 1)^{2/(2_*-2)}\|g\|_{\mathcal{H}'_\mu}. \end{aligned}$$

For each $u \in \mathcal{H}_\mu$, we write

$$t_m := t_{\max}(u) = \left[\frac{\|u\|_{a,\mu}}{(2_* - 1)\int_{\mathbb{R}^N} h\,|y|^{-2_* b}\,|u|^{2_*}\,dx}\right]^{1/(2_*-2)}.$$

Lemma 5. *Let $\lambda \in (0, \Lambda_1)$. For each $u \in \mathcal{H}_\mu$, one has the following:*
(i) If $\int_{\mathbb{R}^N} g(x)\,udx \leq 0$, then there exists a unique $t^- > t_m$ such that $t^- u \in \mathcal{N}^-$ and

$$I\left(t^- u\right) = \sup_{t \geq 0} I(tu).$$

(ii) If $\int_{\mathbb{R}^N} g(x)\,udx > 0$, then there exist unique t^+ and t^- such that $0 < t^+ < t_m < t^-$, $t^+ u \in \mathcal{N}^+$, $t^- u \in \mathcal{N}^-$,

$$I\left(t^+ u\right) = \inf_{0 \leq t \leq t_m} I(tu) \text{ and } I\left(t^- u\right) = \sup_{t \geq 0} I(tu).$$

3. Proof of Theorem 1

For the proof, we recall the following results:

Proposition 2. *(see [4])*
(i) If $\lambda \in (0, \Lambda_1)$, then there exists a minimizing sequence $(u_n)_n$ in $\mathcal{N}$ such that

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1) \text{ in } \mathcal{H}'_\mu, \tag{3.1}$$

where $o_n(1)$ tends to 0 as n tends to ∞.
(ii) if $\lambda \in (0, (1/2)\Lambda_1)$, then there exists a minimizing sequence $(u_n)_n$ in $\mathcal{N}^-$ such that

$$I(u_n) = c^- + o_n(1) \text{ and } I'(u_n) = o_n(1) \text{ in } \mathcal{H}'_\mu.$$

We establish the existence of a local minimum for I on $\mathcal{N}^+$.

Proposition 3. *If $\lambda \in (0, \Lambda_1)$, then I has a minimizer $u_1 \in \mathcal{N}^+$ and it satisfies*
(i) $I(u_1) = c = c^+ < 0$,
(ii) u_1 is a solution of $(\mathcal{P})$.

4. Proof of Theorem 2

In this section, we establish the existence of a second solution of $(\mathcal{P})$. For this, we require the following lemmas with C_0 is given in (1.3).

Lemma 6. *Assume that (G) holds and let $(u_n)_n \subset \mathcal{H}_\mu$ be a $(PS)_c$ sequence for I for some $c \in \mathbb{R}$ with $u_n \rightharpoonup u$ in $\mathcal{H}_\mu$. Then,*

$$I'(u) = 0 \text{ and } I(u) \geq -C_0\lambda^2.$$

Lemma 7. *Assume that (G) holds and for any $(PS)_c$ sequence with c is a real number such that $c < c^*_\lambda$. Then, there exists a subsequence which converges strongly.*

Here $c^*_\lambda := ((2_* - 2)/2_* 2)(h_0)^{-2/(2_*-2)}(S_{\mu,k})^{2_*/(2_*-2)} - C_0\lambda^2$.

Lemma 8. *Assume that (G) and (H) hold. Then, there exist $v \in \mathcal{H}_\mu$ and $\Lambda_* > 0$ such that for $\lambda \in (0, \Lambda_*)$, one has*

$$\sup_{t \geq 0} I(tv) < c_\lambda^*.$$

In particular,

$$c^- < c_\lambda^*, \text{ for all } \lambda \in (0, \Lambda_*).$$

Now we establish the existence of a local minimum of I on $\mathcal{N}^-$.

Proposition 4. *There exists $\Lambda_2 > 0$ such that for $\lambda \in (0, \Lambda_2)$, the functional I has a minimizer u_2 in $\mathcal{N}^-$ and satisfies*

(i) $I(u_2) = c^-$,

(ii) u_2 is a solution of $(\mathcal{P})$ in $\mathcal{H}_\mu$, where $\Lambda_2 = \min\{(1/2)\Lambda_1, \Lambda_*\}$ with Λ_1 defined as in 2.4 and Λ_* defined as in the proof of Lemma 8.

Proof. By Proposition 2 (ii), there exists a $(PS)_{c^-}$ sequence for I, $(u_n)_n$ in $\mathcal{N}^-$ for all $\lambda \in (0, (1/2)\Lambda_1)$. From Lemmas 7, 8 and 4 (ii), for $\lambda \in (0, \Lambda_*)$, I satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Therefore, there exist a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ and $u_2 \in \mathcal{N}^-$ such that u_n converges to u_2 strongly in $\mathcal{H}_\mu$ and $I(u_2) = c^-$ for $\lambda \in (0, \Lambda_2)$. Finally, by using the same arguments as in the proof of the Proposition 3, for all $\lambda \in (0, \Lambda_2)$, we have that u_2 is a solution of $(\mathcal{P})$. □

Now, we complete the proof of Theorem 2. By Propositions 3 and 4, we conclude that $(\mathcal{P})$ has two solutions u_1 and u_2 such that $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that u_1 and u_2 are distinct.

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