Denting points in Orlicz spaces

Adam Bohonos

West Pomeranian University of Technology
Szczecin, Poland

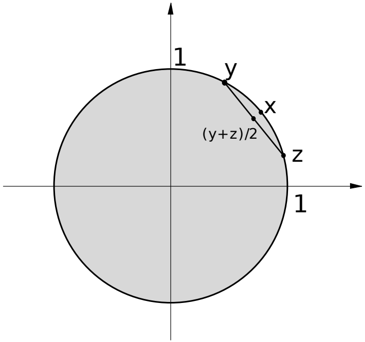
8^{th} International Conference on Function Spaces,
Differential Operators, Nonlinear Analysis
September 18-24, 2011 Tabarz/Thür. (Germany)

Definition

A point $x \in S(X)$ is called an extreme point of the unit ball ($x \in \delta_e B(X)$), if for any $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.

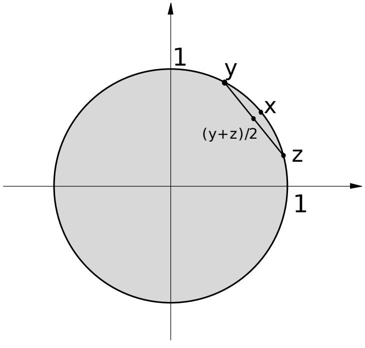
Definition

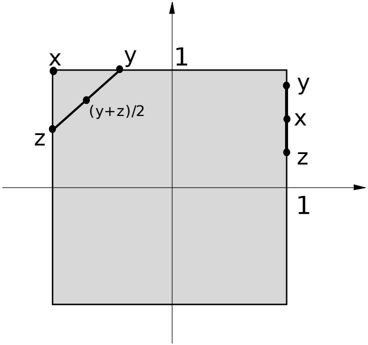
A point $x \in S(X)$ is called an extreme point of the unit ball ($x \in \delta_e B(X)$), if for any $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.



Definition

A point $x \in S(X)$ is called an extreme point of the unit ball $(x \in \delta_e B(X))$, if for any $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.





Definition

A point $x \in S(X)$ is called a strongly extreme point of the unit ball ($x \in \delta_{se}B(X)$), if for any sequences $(y_n), (z_n) \subset B(X)$ we have $\|y_n - x\| \to 0$ whenever $\|y_n + z_n - 2x\| \to 0$.

Definition

A point $x \in S(X)$ is called a denting point of the unit ball ($x \in \delta_d B(X)$), if we have that:

$$x \notin \overline{co}\,\{B(X) \setminus [x + \varepsilon B(X)]\}$$

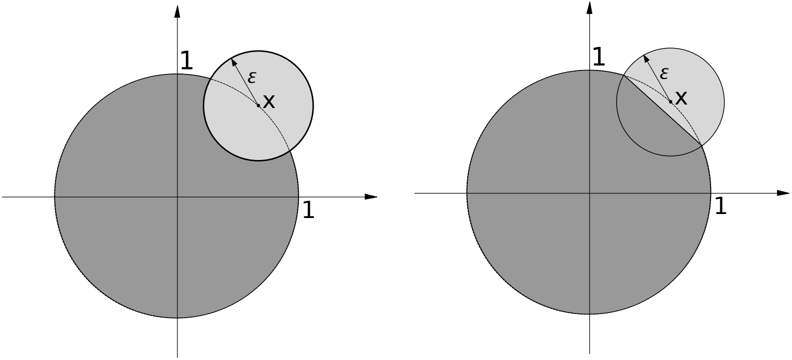
for each $\varepsilon > 0$

Definition

A point $x \in S(X)$ is called a denting point of the unit ball ($x \in \delta_d B(X)$), if we have that:

$$x \notin \overline{co}\,\{B(X) \setminus [x + \varepsilon B(X)]\}$$

for each $\varepsilon > 0$

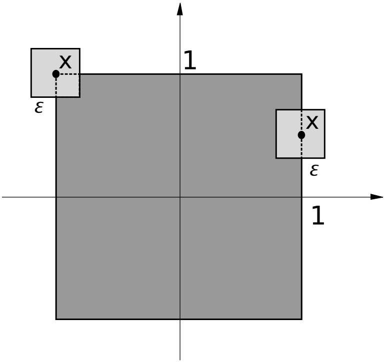


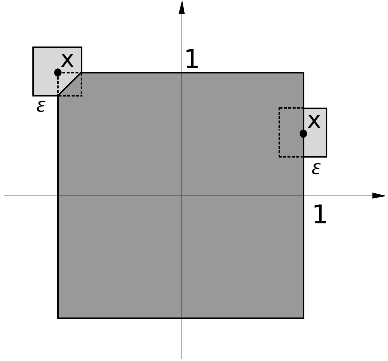
Definition

A point $x \in S(X)$ is called a denting point of the unit ball ($x \in \delta_d B(X)$), if we have that:

$$x \notin \overline{co}\,\{B(X) \setminus [x + \varepsilon B(X)]\}$$

for each $\varepsilon > 0$



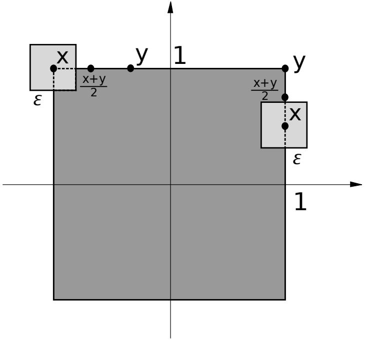


Definition

A point $x \in S(X)$ is called an point of local uniform rotundity of the unit ball ($x \in \delta_{LUR} B(X)$), if for each $\varepsilon > 0$ there is $\delta(x, \varepsilon) > 0$ such that for all $y \in B(X)$ the inequality $\|x - y\| > \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| < 1 - \delta$.

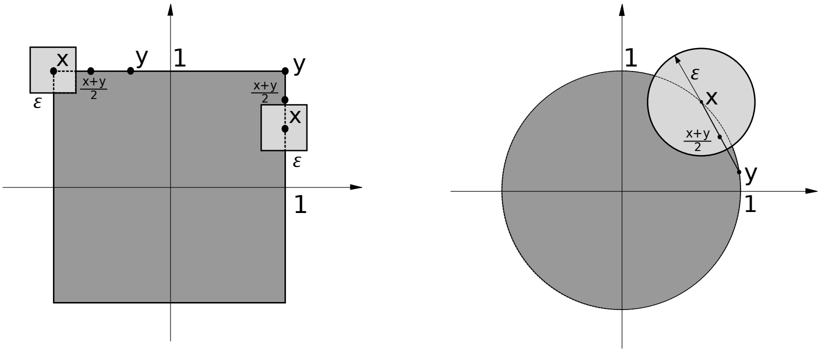
Definition

A point $x \in S(X)$ is called an point of local uniform rotundity of the unit ball ($x \in \delta_{LUR} B(X)$), if for each $\varepsilon > 0$ there is $\delta(x, \varepsilon) > 0$ such that for all $y \in B(X)$ the inequality $\|x - y\| > \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| < 1 - \delta$.



Definition

A point $x \in S(X)$ is called an point of local uniform rotundity of the unit ball ($x \in \delta_{LUR}B(X)$), if for each $\varepsilon > 0$ there is $\delta(x, \varepsilon) > 0$ such that for all $y \in B(X)$ the inequality $\|x - y\| > \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| < 1 - \delta$.



A Banach space X is said to have property

1. **R** ($X \in$ (**R**)) if $S(X) = \delta_e B(X)$,

A Banach space X is said to have property

1. **R** ($X \in$ (**R**)) if $S(X) = \delta_e B(X)$,
2. **MLUR** ($X \in$ (**MLUR**)) if $S(X) = \delta_{se} B(X)$,

A Banach space X is said to have property

1. **R** ($X \in$ (**R**)) if $S(X) = \delta_e B(X)$,
2. **MLUR** ($X \in$ (**MLUR**)) if $S(X) = \delta_{se} B(X)$,
3. **G** ($X \in$ (**G**)) if $S(X) = \delta_d B(X)$,

A Banach space X is said to have property

1. **R** ($X \in$ (**R**)) if $S(X) = \delta_e B(X)$,
2. **MLUR** ($X \in$ (**MLUR**)) if $S(X) = \delta_{se} B(X)$,
3. **G** ($X \in$ (**G**)) if $S(X) = \delta_d B(X)$,
4. **LUR** ($X \in$ (**LUR**)) if $S(X) = \delta_{LUR} B(X)$.

A Banach space X is said to have property

1. **R** ($X \in$ (**R**)) if $S(X) = \delta_e B(X)$,
2. **MLUR** ($X \in$ (**MLUR**)) if $S(X) = \delta_{se} B(X)$,
3. **G** ($X \in$ (**G**)) if $S(X) = \delta_d B(X)$,
4. **LUR** ($X \in$ (**LUR**)) if $S(X) = \delta_{LUR} B(X)$.

Theorem

For any Banach space X we have

$$\delta_{LUR} B(X) \subset \delta_d B(X) \subset \delta_{se} B(X) \subset \delta_e B(X)$$

and

$$LUR \Rightarrow G \Rightarrow MLUR \Rightarrow R.$$

Definition

Given any $x \in S(X)$ a function $\delta_x : [0,2] \to [0,2]$ defined for any $\varepsilon \in [0,2]$ as the distance between x and the set

$$\overline{co}\left\{B(X) \setminus \left[x + \varepsilon B(X)^0\right]\right\}$$

i.e.

$$\delta_x(\varepsilon) = \inf\left\{\|x - y\|_X : y \in \overline{co}\left\{B(X) \setminus \left[x + \varepsilon B(X)^0\right]\right\}\right\}$$

is said to be the modulus of dentability of x.

Definition

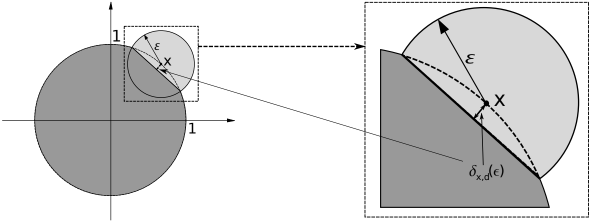
Given any $x \in S(X)$ a function $\delta_x : [0,2] \to [0,2]$ defined for any $\varepsilon \in [0,2]$ as the distance between x and the set

$$\overline{co}\left\{B(X) \setminus \left[x + \varepsilon B(X)^0\right]\right\}$$

i.e.

$$\delta_x(\varepsilon) = \inf\left\{\|x-y\|_X : y \in \overline{co}\left\{B(X) \setminus \left[x + \varepsilon B(X)^0\right]\right\}\right\}$$

is said to be the modulus of dentability of x.



Definition

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if it is even, convex, left continuous on whole of $\mathbb{R}_+$, $\Phi(0) = 0$ and Φ is not identically equal to zero.

Definition

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if it is even, convex, left continuous on whole of $\mathbb{R}_+$, $\Phi(0) = 0$ and Φ is not identically equal to zero.

Definition

We say that an Orlicz function Φ satisfies the $\Delta_2 -$ *condition* for all $u \in \mathbb{R}$ (at infinity) [at zero] if there are positive constants K and u_0 with $0 < \Phi(u_0) < \infty$ such that $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \mathbb{R}$ (for every $|u| \geq u_0$) [for every $|u| \leq u_0$]. We denote these conditions by $\Phi \in \Delta_2$ ($\Phi \in \Delta_2(\infty)$) [$\Phi \in \Delta_2(0)$], respectively. We have that $\Phi \in \Delta_2$ if and only if $\Phi \in \Delta_2(0)$ and $\Phi \in \Delta_2(\infty)$.

Let (T, Σ, μ) be a $\sigma-$finite and complete measure space. By $L^0 = L^0(T)$ we denote the set of all μ-equivalence classes of real valued measurable functions defined on T.
For any Orlicz function Φ we can define on $L^0(T)$ a convex functional

$$I_\Phi(x) = \int_T \Phi(x(t))d\mu.$$

We define the Orlicz space L_Φ generated by an Orlicz function Φ by the formula

$$L_\Phi = \left\{x \in L^0(T) : I_\Phi(cx) < \infty \texttt{ for some } c > 0 \texttt{ depending on } x\right\}.$$

In this paper we will use so called Orlicz norm with can be written in the Amemiya formula

$$\|x\|_\Phi^0 = \inf_{k>0} \frac{1}{k}\left(1 + I_\Phi(kx)\right)$$

Theorem (Y. Cui, H. Hudzik, R. Płuciennik 2003)

Let Φ be an arbitrary Orlicz function. Then $x \in S(L^0_\Phi)$ is an extreme point of the unit ball $B(L^0_\Phi)$ if and only if

1. *The set $K(x)$ consists of one element ($K(x) = \{k\}$, where $k > 0$).*
2. *$kx(t) \in SC(\Phi)$ for $\mu -$ a.e. $t \in T$.*

$K(x)$ - is the set of x for which the infimum of definition in the Amemiya formula is attained. $SC(\Phi)$ - is a set of extreme points of epigraph of Φ.

Theorem (Y. Cui, H. Hudzik, R. Płuciennik 2003)

Let Φ be an arbitrary Orlicz function. Then $x \in S(L^0_\Phi)$ is an strongly extreme point of the unit ball $B(L^0_\Phi)$ if and only if and only if the following conditions are satisfied:

1. *The set $K(x)$ consists of one element ($K(x) = \{k\}$, where $k > 0$).*
2. *$kx(t) \in SC(\Phi)$ for $\mu -$ a.e. $t \in T$.*
3. *Either $\Phi(b(\Phi)) < \infty$ and x is of the form $k|x(t)| = b(\Phi)$ for $\mu -$ a.e. $t \in T$, or $\Phi \in \Delta_2(\infty)$ and at least one of the conditions*
   - i) $\mu(T) < \infty$
   - ii) $a(\Phi) > 0$
   - iii) $\Phi \in \Delta_2(0)$.

$a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\}$
$b(\Phi) = \sup\{u > 0 : \Phi(u) < \infty\}$

Theorem

Let Φ be an Orlicz function. If $b(\Phi) < \infty$, then the set of denting points of $B(L^0_\Phi)$ is empty.

Theorem

Let Φ be an Orlicz function. If $b(\Phi) < \infty$, then the set of denting points of $B(L^0_\Phi)$ is empty.

Theorem

Let Φ be an Orlicz function and $\mu(T) = \infty$. If $a(\Phi) > 0$, then the set of denting points of $B(L^0_\Phi)$ is empty.

Theorem

Assume Φ is an Orlicz function and $x \in S(L^0_\Phi)$. If x is a denting point of $B(L^0_\Phi)$, then the following conditions are satisfied:

1. *The set $K(x)$ is a singleton;*
2. *$kx(t) \in SC(\Phi)$ for μ-a.e. $t \in T$;*
3. *$\Phi \in \Delta_2(\infty)$ and at least one of the conditions holds:*
   - (i) $\mu(T) < \infty$;
   - (ii) $\Phi \in \Delta_2(0)$.

Example

Consider the space $L^1 \cap L^\infty(T)$ equipped with norm:

$$\|x\|_{L^1 \cap L^\infty} = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty} .$$

It is Orlicz space with Orlicz norm generated by the Orlicz function

$$\Phi(u) = \begin{cases} u & \textit{for} \quad |u| \le 1 \\ \infty & \textit{for} \quad |u| > 1 \end{cases} .$$

- If $\mu(T) = \infty$ then $\delta_d B(L^1 \cap L^\infty) = \delta_{se} B(L^1 \cap L^\infty) = \varnothing \neq \delta_e B(L^1 \cap L^\infty)$.
- If $\mu(T) < \infty$ then $\delta_d B(L^1 \cap L^\infty) = \varnothing \neq \delta_{se} B(L^1 \cap L^\infty) = \delta_e B(L^1 \cap L^\infty)$.

$L^1 + L^\infty$

Consider the classical interpolation space $L^1 + L^\infty(T)$ equipped with natural norm:

$$\|x\|_{L^1+L^\infty} = \inf\left\{\|y\|_1 + \|z\|_\infty : y + z = x,\ y \in L^1,\ z \in L^\infty\right\}.$$

It is Orlicz space with Orlicz norm generated by the Orlicz function

$$\Phi(u) = \begin{cases} 0 & \textit{for} \quad |u| \leq 1 \\ |u| - 1 & \textit{for} \quad |u| > 1 \end{cases}.$$

Theorem

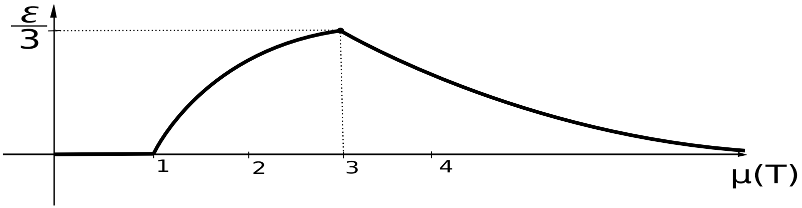
a) *The set* $\delta_d B\left(L^1 + L^\infty\right)$ *is non-empty if and only if* $1 < \mu(T) < \infty$.

b) *Let* $1 < \mu(T) < \infty$. *A point* $x \in S\left(L^1 + L^\infty\right)$ *is a denting point if and only if* $|x| = \chi_T$. *Moreover, for any* $\varepsilon \in (0,1)$ *and any* $x \in \delta_d B\left(L^1 + L^\infty\right)$ *the modulus of dentability* $\delta_x(\varepsilon) = \min\left\{\frac{(\mu(T)-1)\varepsilon}{2\mu(T)}, \frac{\varepsilon}{\mu(T)}\right\}$.

Theorem

a) *The set* $\delta_d B\left(L^1 + L^\infty\right)$ *is non-empty if and only if* $1 < \mu(T) < \infty$.

b) *Let* $1 < \mu(T) < \infty$. *A point* $x \in S\left(L^1 + L^\infty\right)$ *is a denting point if and only if* $|x| = \chi_T$. *Moreover, for any* $\varepsilon \in (0,1)$ *and any* $x \in \delta_d B\left(L^1 + L^\infty\right)$ *the modulus of dentability* $\delta_x(\varepsilon) = \min\left\{\frac{(\mu(T)-1)\varepsilon}{2\mu(T)}, \frac{\varepsilon}{\mu(T)}\right\}$.



Corollary

In the space $L^1 + L^\infty$ with norm $\|x\|_{L^1+L^\infty}$ the only denting points are

$$|x(t)| = \chi_T.$$

and

- *If $\mu(T) = \infty$ then* $\delta_d B(L^1 + L^\infty) = \varnothing \neq \delta_{se} B(L^1 + L^\infty) = \delta_e B(L^1 + L^\infty)$.
- *If $1 < \mu(T) < \infty$ then* $\delta_d B(L^1 + L^\infty) = \delta_{se} B(L^1 + L^\infty) = \delta_e B(L^1 + L^\infty) \neq \varnothing$.
- *If $\mu(T) \leq 1$ then* $\delta_d B(L^1 + L^\infty) = \delta_{se} B(L^1 + L^\infty) = \delta_e B(L^1 + L^\infty) = \varnothing$.

Corollary

In the space $L^1 + L^\infty$ *with norm* $\|x\|_{L^1+L^\infty}$ *the only denting points are*

$$|x(t)| = \chi_T.$$

and

- *If* $\mu(T) = \infty$ *then* $\delta_d B(L^1 + L^\infty) = \varnothing \neq \delta_{se} B(L^1 + L^\infty) = \delta_e B(L^1 + L^\infty)$.
- *If* $1 < \mu(T) < \infty$ *then* $\delta_d B(L^1 + L^\infty) = \delta_{se} B(L^1 + L^\infty) = \delta_e B(L^1 + L^\infty) \neq \varnothing$.
- *If* $\mu(T) \leq 1$ *then* $\delta_d B(L^1 + L^\infty) = \delta_{se} B(L^1 + L^\infty) = \delta_e B(L^1 + L^\infty) = \varnothing$.

Theorem

$\delta_{LUR} B(L^1 + L^\infty) = \phi$.

All results presented here was obtained in collaboration with prof. Ryszard Płuciennik from Poznań University of Technology.

Thank for your attention