Characterization of Spaces of Functions of Zero Smoothness

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For functions of many variables in the spaces $B^s_{p,\theta}(\mathbb{R}^n)$, $0 \le s < 1$, we study new difference characteristics which define equivalent norms for $0 < s < 1$, as well as for $s = 0$ in some cases. Using the averaged difference of a function f,

$$\delta(h)f(x) := (2h)^{-2n} \int_{[-h,h]^n} \int_{[-h,h]^n} |f(x+y) - f(x+z)| \, dy \, dz,$$

we construct Banach spaces $\overline{B}^s_{p,\theta}(\mathbb{R}^n)$, $0 \le s < 1$.

Here $\mathbb{R}^n$ is the Euclidean n-space of points $x = (x_1, \dots, x_n)$, $1 \le p \le \infty$.

Let L_p be the Lebesgue space with the norm $\|f|L_p\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}$.

Consider the Banach space $B^s_{p,\theta} = B^s_{p,\theta}(\mathbb{R}^n)$ of generalized functions $f \in S'$ with the norm

$$\|f|B^s_{p,\theta}\| = \left\{ \sum_{j=0}^{\infty} 2^{s\theta} \|a_j|L_p\|^\theta \right\}^{1/\theta}, \qquad s \in \mathbb{R}, \quad 1 \le p \le \infty, \tag{1}$$ eq1

where

$$a_j(x) = F^{-1}\varphi_j Ff, \qquad \varphi_0(x) = \varphi(x), \qquad \varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x),$$

$$\varphi \in C_0^\infty(\mathbb{R}^n), \qquad \varphi(x) = 1 \quad \text{for} \quad |x| \le 1, \qquad \varphi(x) = 0 \quad \text{for} \quad |x| \ge 2,$$

so that

$$\operatorname{supp} \varphi_0 \subset \{x : |x| \le 2\},$$

$$\operatorname{supp} \varphi_j \subset \{x : 2^{j-1} \le |x| \le 2^{j+1}\} \quad \text{for} \quad j \in \mathbb{N},$$

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \forall x \in \mathbb{R}^n,$$

$$\varphi_j(x) = \varphi_1\Big(\frac{x}{2^{j-1}}\Big), \qquad \varphi_1(x) = \varphi\Big(\frac{x}{2}\Big) - \varphi(x).$$

We compare the space $B^s_{p,\theta}$, $0 \le s < 1$, with four other Banach function spaces defined for $0 \le s < 1$, $1 \le p \le \infty$, and $1 \le \theta \le \infty$ as the spaces of locally integrable functions on $\mathbb{R}^n$ with the norms

$$\|f|^\Delta B^s_{p,\theta}\| := |||f|||_{p,s} + \left\{ \int_{|y|<1} \left(\frac{\|\Delta(y)f|L_p\|}{|y|^s} \right)^\theta \frac{dy}{|y|^n} \right\}^{1/\theta}, \tag{2}$$ eq2

$$\|f|{}^{\overline{\Delta}}B^s_{p,\theta}\| := |||f|||_{p,s} + \left\{\int_0^1 \left(\frac{\|\overline{\Delta}(h)f|L_p\|}{h^s}\right)^\theta \frac{dh}{h}\right\}^{1/\theta}, \qquad (3) \quad \boxed{\texttt{eq3}}$$

where $\Delta(y)f(x) = f(x+y) - f(x)$, $\overline{\Delta}(h)f(x) = h^{-n}\int_{|y|<h} |\Delta(y)f(x)|\,dy$, $Q = [-1,1]^n$,

$$|||f|||_{p,s} := \begin{cases} \|f|L_p\| & \text{for } 0 < s < 1,\ 1 \le p \le \infty; \\ \left\|\int_Q |f(\cdot + y)|\,dy\Big|L_p\right\| & \text{for } s = 0,\ 1 \le p \le \infty. \end{cases}$$

It is known that for $0 < s < 1$ the norms (1), (2) and (3) are equivalent.

$$\|f|\overline{B}^s_{p,\theta}\| := |||f|||_{p,s} + \left\{\int_0^1 \left(\frac{\|\delta(h)f|L_p\|}{h^s}\right)^\theta \frac{dh}{h}\right\}^{1/\theta}, \qquad (4) \quad \boxed{\texttt{eq4}}$$

where

$$\delta(h)f(x) := (2h)^{-2n}\int_{[-h,h]^n}\int_{[-h,h]^n} |f(x+y) - f(x+z)|\,dy\,dz,$$

$$\|f|\widetilde{B}^s_{p,\theta}\| := |||f|||_{p,s} + \left\{\int_0^1 \left(\frac{\|v_h * f|L_p\|}{h^s}\right)^\theta \frac{dh}{h}\right\}^{1/\theta}, \qquad (5) \quad \boxed{\texttt{eq5}}$$

where

$$v_h = h^{-n}v\Big(\frac{x}{h}\Big), \qquad v(x) = 2\omega\Big(\frac{x}{2}\Big) - \omega(x), \qquad \omega \in C_0^\infty(\mathbb{R}^n),$$

$$\operatorname{supp}\omega \subset Q, \qquad \omega(x) = \overline{\omega}(|x|), \qquad \int \omega(x)\,dx \neq 0.$$

Theorem 1. *For* $0 < s < 1$

$${}^{\Delta}B^s_{p,\theta} = {}^{\overline{\Delta}}B^s_{p,\theta} = \overline{B}^s_{p,\theta} = \widetilde{B}^s_{p,\theta} = B^s_{p,\theta}.$$

For $s = 0$

$${}^{\Delta}B^0_{p,\theta} \subset {}^{\overline{\Delta}}B^0_{p,\theta} \subset \overline{B}^0_{p,\theta} \subset \widetilde{B}^0_{p,\theta} \subset B^0_{p,\theta},$$

$$\overline{B}^0_{p,\theta} \neq \widetilde{B}^0_{p,\theta} \qquad \textit{for} \quad 1 \le \theta < \infty.$$

For comparison, consider the space $\mathrm{bmo}(\mathbb{R}^n) = F^0_{\infty,2}(\mathbb{R}^n)$ of locally integrable functions with the norm

$$\|f|\mathrm{bmo}(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n}\int_Q |f(x+y)|\,dy + \sup_{h>0,\, x\in R^n}(2h)^{-n}\int_{hQ} |f(x+y) - f_h(x)|\,dy, \quad (6)$$

where $f_h(x) = (2h)^{-n}\int_{hQ} f(x+y)\,dy$, $hQ = [-h,h]^n$.

Lemma 1.

$$\overline{B}^0_{\infty,\infty}(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n) = F^0_{\infty,2} \subset \widetilde{B}^0_{\infty,\infty}(\mathbb{R}^n).$$

It is interesting to find out when $\tilde{B}^0_{p,\theta}$ and $B^0_{p,\theta}$ coincide.

This problem is solved by using the following Sickel–Triebel result (1995):

Let $1 \le p \le \infty$ and $1 \le \theta \le \min\{p, 2\}$. Then

$$B^0_{p,\theta} \subset \begin{cases} L_p & \text{for } p < \infty, \\ \text{bmo} & \text{for } p = \infty. \end{cases}$$

Theorem 2. *For* $1 \le p \le \infty$ *and* $1 \le \theta \le \min\{p, 2\}$

$$\widetilde{B}^0_{p,\theta}(\mathbb{R}^n) = B^0_{p,\theta}(\mathbb{R}^n).$$

In $\mathbb{R}^n$, consider domains satisfying the flexible cone condition. The definition of the space $\overline{B}^0_{p,\theta}(\mathbb{R}^n) = \overline{B}^0_{p,\theta}$ extends naturally to such domains.

On a domain G satisfying the flexible cone condition, consider the spaces $B^s_{p,\theta}(G)$, $s > 0$, and the Sobolev spaces $W^s_p(G)$, $s \in \mathbb{N}$,

$$\|f|W^s_p(G)\| = \|f|L_p\| + \sum_{i=1}^{n} \|D^s_i f|L_p(G)\|.$$

t3 **Theorem 3.** *Let G be a domain satisfying the flexible cone condition,* $1 \le p < q \le \infty$, $s = \frac{n}{p} - \frac{n}{q}$ *and* $1 \le \theta \le \infty$. *Then*

$$B^s_{p,\theta}(G) \subset \overline{B}^0_{q,\theta}(G).$$

t4 **Theorem 4.** *Let G be a domain satisfying the flexible cone condition,* $1 \le p < q \le \infty$, *and* $s = \frac{n}{p} - \frac{n}{q}$. *Then*

$$W^s_p(G) \subset \overline{B}^0_{q,p}(G) \qquad \textit{for } \; s \in \mathbb{N}, \;\; p > 1.$$

Remark. If $G = \mathbb{R}^n$, $1 < p \le 2$, $2 \le q \le \infty$ the statement of Theorem 4 strengthens the known embeddings

$$W^s_p \subset L_q \qquad \text{for} \quad q < \infty,$$

$$W^s_p \subset \text{bmo} \qquad \text{for} \quad q = \infty.$$

The proofs are based on integral representations of functions in terms of their derivatives and differences.