Some Fixed Points Results on CAT(0) Spaces

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First Part, Review

Historical notes

More recently, many of the standard ideas of nonlinear analysis have been extended to the class of so-called *CAT(0)* spaces,[*So named by Gromov in honor of Cartan, Alexandrov, and Toponogov*]. First time, W.A. Kirk developed the fixed point theory for CAT(0) spaces and proved an interesting fact about the fixed point set. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed. In 2008, Kirk and Panyanak used the concept of $\triangle$-convergence introduced by Lim to prove the CAT(0) space analogs of some Banach space results which involve weak convergence and Dhompongsa and Panyanak obtained $\triangle$-convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting.

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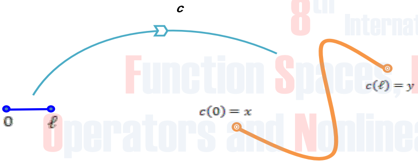
Basic Definitions

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, \ell] \subseteq \mathbb{R}$ to X such that $c(0) = x, c(\ell) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, \ell]$. In particular, c is an isometry and $d(x, y) = \ell$. The image of c is called a geodesic (or metric) segment joining x and y. When it is *unique*, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.



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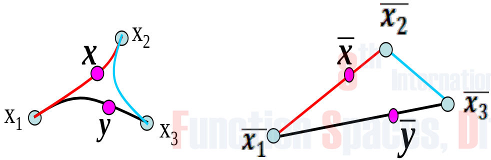
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

"Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})."$$



Let $\{x_n\}$ be a bounded sequence in X and K be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\},$$

where

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x),$$

and the set

$$A = A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to K. Similarly, set A is called the *asymptotic center* of $\{x_n\}$ relative to K.

In the CAT(0) space, the asymptotic center $A = A(K, \{x_n\})$ of $\{x_n\}$ consists of exactly one point whenever K is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \neq y$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

So every CAT(0) space X satisfies the Opial property.

Lemma

Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.([10])

Lemma

If K is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of is in K.([11])

Definition

A metric space is hyperconvex if every family of balls $\{B(x_i, r_i)\}_{i\in I}$ that has $d(x_i, x_j) \le r_i + r_j$ for all i, j has a nonempty intersection $\bigcap_{i\in I} B(x_i, r_i) \ne \emptyset$.

For example, the unit ball of ℓ_∞ is a complete bounded hyperconvex metric space.

Definition

A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is such that

(W1) $d(z, W(x, y, t)) \le (1 - t)d(z, x) + td(z, y)$

(W2) $d(W(x, y, t), W(x, y, s)) = |t - s|d(x, y)$

(W3) $W(x, y, t) = W(y, x, 1 - t)$

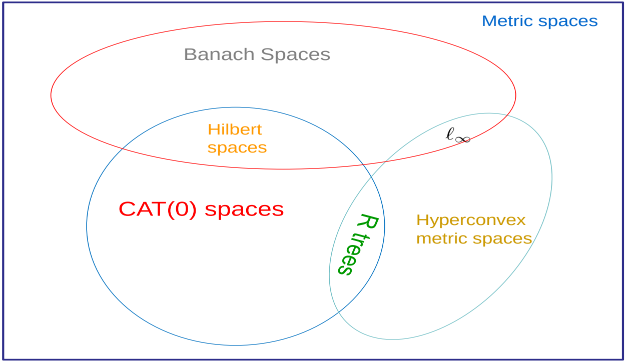
(W4) $d(W(x, z, t), W(y, w, t)) \le (1 - t)d(x, y) + td(z, w)$

for all $x, y, z, w \in X$ and $t, s \in [0, 1]$.

Definition

A $\mathbb{R}$-tree is a nonempty metric space X satisfying:

1. Any two points $x, y \in X$ are the endpoints of a unique metric segment $[x, y]$.
2. If $x, y, z \in X$ then $[x, y] \cap [x, z] = [x, w]$ for some $w \in X$ (i.e., if we have two metric segments with a common endpoint, then their intersection is a metric segment.)
3. If $x, y, z \in X$ and $[x, y] \cap [y, z] = \{y\}$ then $[x, y] \cup [y, z] = [x, z]$ (i.e., if two segments intersect in a single point, then their union is a metric segment.)



Hilbert spaces (in which the CAT(0) inequality is an equality); the only Banach spaces that are CAT(0).
$\mathbb{R}$-trees; the only hyperconvex metric spaces that are CAT(0).

Dhompongsa and Panyanak's Results

Comp. and Math. with Appl., 2008

Lemma

Let (X, d) be a CAT(0) space. Then

1. *Let $x, y \in X, \alpha \in [0,1]$, $m \in [p, x]$ and $m_2 \in [p, y]$ satisfying*

$$d(m_1, m_2) = \alpha d(p, x), \quad d(p, m_2) = \alpha d(p, y).$$

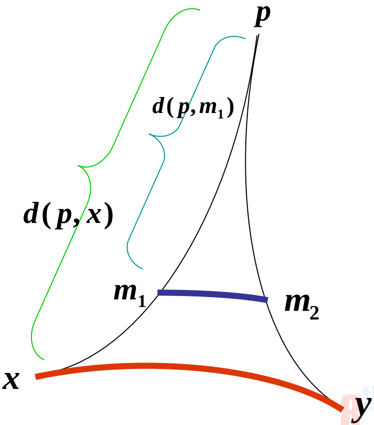
Then $d(m_1, m_2) \leq \alpha d(x, y)$.

2. *Let $x, y \in X$. For $t \in [0,1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y), \quad d(y, z) = (1-t)d(x, y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point z.

Illustration



$$\frac{d(p,m_1)}{d(p,x)} = \frac{d(p,m_2)}{d(p,y)} = \alpha \in [0,1] \Rightarrow \frac{d(m_1,m_2)}{d(x,y)} \le \alpha$$

Chaoha and Phon-on's Results

J. Math. Anal. Appl., 2006

Theorem

Let A be a nonempty subset of a CAT(0) space (X, d). Then there exists a continuous map $f : X \to X$ such that $Fix(f) = \bar{A}$.

Note. Fix $x_0 \in A$ and define
$f(x) = (1 - \frac{d(x,A)}{1+d(x,A)})x \oplus \frac{d(x,A)}{1+d(x,A)}x_0$, for all $x \in X$.

Some Results

Lemma
Let (X, d) *be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z) \leq \max\{d(x,z), d(y,z)\},$$

for $x, y, z \in X$ *and* $t \in [0,1]$.

Lemma
Let (X, d) *be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2,$$

for all $x, y, z \in X$ *and* $t \in [0,1]$.

Some Results

In particular by Lemma 9, $t := \frac{1}{2}$, we have

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y)^2 \leq \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2,$$

for all $x, y, z \in X$, which is called (CN) inequality of Bruhat-Tits[1]. In fact[2],
a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

[1] as it was shown in [3]
[2] (cf. [4], p. 163)

Second Part

Historical and Review

In 2008 T. Suzuki [[5], T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl., V 340, 2008, 1088-1095.] defined *condition (C)* for a mapping on subset of Banach space, as following: "Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy *condition (C)* if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$."

This condition was weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In that paper, he presented fixed point theorems and convergence theorems for mappings satisfying condition (C). Also the examples 1 and 2 in same paper stated that there exists a map T which satisfies condition (C), but T is not nonexpansive, and there exists a map T which is quasi-nonexpansive, but it does not satisfy condition (C).

Review and Definition

Recently B. Nanjaras, B. Panyanaka and W. Phuengrattana in [6], A. Razani and H. Salahifard in [7] and other mathematicians proved some theorems according to single-valued mapping or multi-valued mapping, which are satisfying Suzuki's condition (C), in a CAT(0) spaces.

Definition

([5]) Let T be a mapping on a subset K of a CAT(0) space (X, d). Then T said to satisfy *condition (C)* if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in K$.

Definition

The following we used this notation $T_\alpha$where $T_1, \cdots, T_n$ are selfmaps on a subset K of X and $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$.

Definition

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ be a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$. The maps $T_1, \cdots, T_n$ on X are said to be α-nonexpansive if

$$\sum_{i=1}^{n} \alpha_i d(T_i x, T_i y) \leq d(x, y), \tag{1}$$

for all $x, y \in X$.([9])

Remarks

Remark. Some observations are immediate. First: the condition (1) implies that $T_1, \cdots, T_n$ are α-nonexpansive and the mapping T_α is nonexpansive when T_1, T_2 are nonexpansive in the hyperbolic spaces (X, d) because we have

$$\begin{aligned} d(T_\alpha x, T_\alpha y) &= d(\alpha_1 T_1 x \oplus \alpha_2 T_2 x, \alpha_1 T_1 y \oplus \alpha_2 T_2 y) \\ &\leq \sum_{i=1}^{2} \alpha_i d(T_i x, T_i y) \leq d(x, y), \end{aligned}$$

for every $x, y \in X$. However, the condition (1) is stronger. Second: From α-nonexpansiveness of T_α we have $\alpha_i d(T_i x, T_i y) \leq d(x, y)$ for all $1 \leq i \leq n$ so, if $\alpha_j = 1$ for some $1 \leq j \leq n$ then T_j is nonexpansive, in which case $T_\alpha = T_j$, and if $\alpha_i \neq 0$ for $1 \leq i \leq n$, then T_i is Lipschitz thus it is uniformly continuous. Third: all nonexpansive mappings satisfy (1). Fourth: for a simple case of the previous definition we can consider the following definition for a map.
Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$ be a multiindex satisfying $\sum_{i=1}^{n} \alpha_i = 1$. A mapping $T : X \to X$ is said to be α-nonexpansive if

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^i y) \leq d(x, y), \tag{2}$$

for all $x, y \in X$.

Example

Example

Let $X = \ell^1$ with $d(x, y) = \|x - y\|_{\ell^1}$ and B be its unit ball. Define the mapping $T : B \to B$ by

$$Tx = T(x_1, x_2, x_3, \cdots) = (x_2^2, ax_3, x_4, \cdots),$$

where $a = \frac{\sqrt{17}-1}{4}$. T is α-nonexpansive with $\alpha = (\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$. The Lipschitz constant $k(T)$ is 2 and $k(T^2) = 2a^2 > 1$. For further iterations,

$$T^n x = (a^2 x_{n+1}^2, ax_{n+2}, x_{n+3}, \cdots), \quad n \geq 3.$$

All have the same Lipschitz constant as T^2, $k(T^n) = k(T^2) > 1$. For more details refer [9].

Our Results

Theorem

Let (X, d) be a complete CAT(0) space. If T, S be two self map on X with $F(T)\bigcap F(S) \neq \emptyset$ and $U := (1-t)T \oplus tS$ a self map on X for every $t \in [0,1]$. And also T_α which $T_1, \cdots, T_n$ are selfmaps on X $(1 \leq i \leq n)$, then

1. *There exists a continuous map T' on X such that $F(T') = F(T)\bigcap F(S)$.*
2. *$d(T'x, Ux) \leq (1-t)d(T'x, Tx) + td(T'x, Sx)$, for every $x \in X$.*
3. *$d(Ux, Uy) \leq (1-t)d(Tx, Ty) + td(Sx, Sy)$, for every $x, y \in X$ and more $F(T') \subseteq F(U)$.*
4. *Further, if T, S be nonexpansive self maps on X, then U is too and $F(U) \subseteq F(T')$.*
5. *There exists a continuous map T' on X such that $F(T') = \bigcap_{i=1}^{n} F(T_i)$.*
6. *We have*
$$d(T'x, T_\alpha x) \leq \sum_{i=1}^{n} \alpha_i d(T'x, T_i x),$$
for every $x \in X$.
7. *$F(T') \subseteq F(T_\alpha)$.*
8. *If T_i be nonexpansive selfmaps on X for $(1 \leq i \leq n)$, then T_α is too, and $F(T_\alpha) \subseteq F(T')$.*

Our Results

World Applied Sciences Journal, 2010

Lemma

Let (X, d) *be a CAT(0) space,* $F := Fix(T)$ *and*

$$\mathcal{F} := \{Fix(T) | T : X \to X,\ \emptyset \neq Fix(T) \text{ and closed}\} \cup \{\emptyset, X\},$$

then

a) *If* $F_\alpha \in \mathcal{F}$ *for every* $\alpha \in I$, *then* $\bigcap_{\alpha \in I} F_\alpha \in \mathcal{F}$.

b) *If* $F_i \in \mathcal{F}$ *for* $1 \leq i \leq n$, *then* $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$.

Lemma

If $\mathcal{T} := \{F | F^c \in \mathcal{F}\}$, *Then* $\mathcal{T}$ *is a topology on* X.

Our Results

Theorem

Let K be a bounded closed convex subset of a complete CAT(0) space (X, d). If $T_\alpha : K \to K$ defined by T_α which $T_1, \cdots, T_n$ are selfmaps on K, which commute each other and satisfying condition (C). Then T_α has fixed point.

Theorem

Let K be a bounded closed convex subset of a complete CAT(0) space (X, d). If $T_\alpha : K \to K$ defined by $T_1, \cdots, T_n$ which are selfmaps on K, which T_1 satisfies the condition (C) and $d(x, T_n x) \leq d(x, T_1 x)$ for every $x \in K$. Then $\inf_{x \in K} d(x, T_\alpha x) = 0$.

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Our Results

Corollary

Let K be a bounded closed convex subset of a complete CAT(0) space (X, d). If $T : K \to K$ satisfies the condition (C), then there exists an approximate fixed point sequence for T, i.e., $\inf_{x\in K} d(x, Tx) = 0$.([7, Lemma 2.5])

References

L. Leustean,A quadratic rate of asymptotic regularity for CAT(0)-spaces, J. Math. Anal. Appl., V 325, 2007, 386399.

S. Dhompongsa and B. Panyanak, On $\triangle$-convergence theorems in CAT(0) spaces, Comp. and Math. with Appl., V 56, 2008, 2572-2579.

F. Bruhat and J. Tits, Groupes ŕeductifs sur un corps local. I. Données radicielles valuées, Inst. Hautes Études Sci. Publ. Math. 41 5-251, (1972).

M. Bridson and A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl., V 340, 2008, 1088-1095.

B. Nanjaras and B. Panyanaka and W. Phuengrattana, Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces, Nonlinear Analysis: Hybrid Systems, V 4, 2010, 25-31.

A. Razani and H. Salahifard, Invariant approximation for CAT(0) spaces, Nonlinear Analysis: Theory, Methods and Applications, V 72, 2010, 2421-2425.

P. Chaoha and A. Phon-on, A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl., V 320, 2006, 983-987.

K. Goebel and W. A. Kirk, Some problems in metric fixed point theory, J. of fixed point theory and applications, V 4, 2008, 13-25 .

W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Anal.*, **68**(2008) 36893696.

S. Dhompongsa, W.A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, *J. Nonlinear Convex Anal.*, **8**(2007) 3545.

THANK YOU

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