On extrapolation results of Yano's type

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- We shall work in the measure space $(\mathbb{R}_+, v)$ with v a positive and locally function in $\mathbb{R}_+$ called weight. We write

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- Given a weight v in $\mathbb{R}_+$ we denote with V its primitive

$$V(t) = \int_0^t v(s)ds.$$

In 1951, Yano using the ideas of Titchmarsh (1929) proved that for every sublinear operator T satisfying that

$$\left(\int_{\mathcal{N}} |Tf(x)|^p d\nu(x)\right)^{\frac{1}{p}} \leq \frac{C}{p-1}\left(\int_{\mathcal{M}} |f(x)|^p d\mu(x)\right)^{\frac{1}{p}},$$

where $\mathcal{N}$, $\mathcal{M}$ are two finite measure spaces for every $1 < p \leq p_0$ it holds that

$$T : L\log L(\mu) \to L^1(\nu)$$

is bounded, where $L\log L(\mu)$ consists of all μ-measurable functions f on $\mathcal{M}$ for which

$$\int_{\mathcal{M}} |f(x)|(1+\log^+|f(x)|)d\mu(x) < \infty.$$

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- and in fact

$$T : L \log L(\mu) \to L^1(\nu) + L^\infty(\nu)$$

is bounded.

More recently the work by N.Y. Antonov [1996], *Convergence of Fourier series* has stimulated new research on endpoint estimates (M.J. Carro and J. Martin [2002], *Extrapolation theory for the real interpolation method*; P. Sjölin and F. Soria [2003], *Remarks on a theorem by N.Y. Antonov*; M.J. Carro and J. Martin [2004], *Endpoint estimates from restricted rearrangement inequalities*; M.J. Carro [2004], *From restricted weak type to strong type estimates*) and has brought new ideas to this old theory.

Theorem (M.J. Carro, 2000)

If μ and ν are two σ-finite measures and T satisfies that

$$T : L^{p,1}(\mu) \to L^{p,\infty}(\nu)$$

is bounded for every $1 < p < p_0$ with constant $\frac{C}{p-1}$, where $L^{p,\infty}(\nu)$ is endowed with the norm

$$\|f\|_{L^{p,\infty}} = \sup_t (t^{\frac{1}{p}} f_\nu^{**}(t))$$

then

$$T : L \log L(\mu) \to M(\phi; \nu).$$

Here $\phi(t) = \frac{t}{1+\log^+ t}$ and $M(\phi) = M(\phi;\nu)$ is the maximal Lorentz space defined as the set of measurable functions such that

$$\|f\|_{M(\phi)} = \sup_{t>0} \phi(t) f_\nu^{**}(t) < \infty,$$

where

$$f_\nu^{**}(t) = \frac{1}{t}\int_0^t f_\nu^*(s)ds$$

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and f_ν^* is the decreasing rearrangement of f with respect to the measure ν. It turns out that this space $M(\phi)$ is strictly embedded in $L^1(\nu) + L^\infty(\nu)$, for every $1 < p < \infty$ and therefore, Yano's theorem was improved.

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Theorem (A. Zygmund)

If T is a linear operator satisfying

$$\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)},$$

for every p near ∞ and μ, ν are finite measures, then

$$T : L^\infty(\mu) \to L_{exp}(\nu),$$

where L_{exp} consists of all ν-measurable functions f for which there is a constant $\lambda = \lambda(f) > 0$ such that $\int_{\mathcal{M}} \exp(\lambda|f(x)|)dx < \infty$.

Using a duality argument this result was recently extended by M. J. Carro to the case of general measures

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$$\sup_{t>0} \frac{(Tf)^{**}_{\nu}(t)}{1+\log^+ \frac{1}{t}} \leq C\left(\int_1^\infty f^{**}_{\mu}(t)\frac{dt}{t} + \|f\|_\infty\right).$$

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M.J. Carro and J. Martin in 2002 have presented an alternative extrapolation theory for the real interpolation method. Their method has the advantage, among others, of obtaining a better range space in the case of Yano's type result and a better domain in the case of Zygmund's type results than the ones known up to that time.

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has become very active in the last two decades.

- A starting point of this theory was the characterization of the boundedness of the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

in weighted Lorentz spaces $\Lambda^p(w)$ defined by the condition

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t) dt \right)^{\frac{1}{p}} < \infty.$$

It was proved by M. A. Ariño and B. Muckenhoupt in 1990 that for every $p > 1$

$$M : \Lambda^p(w) \to \Lambda^p(w)$$

is bounded if and only if $w \in B_p$; that is

$$\|w\|_{B_p} = \sup_{r>0} \frac{\int_0^r w(t)dt + r^p \int_r^\infty \frac{w(t)}{t^p} dt}{\int_0^r w(t)dt} < \infty.$$

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Since

$$(Mf)^*(t) \approx f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds,$$

the above boundedness can be described in terms of the Hardy operator on the cone of decreasing functions.

In fact, it holds that, for every f positive and decreasing

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)ds\right)^p w(t)dt\right)^{\frac{1}{p}} \leq A\left(\int_0^\infty f(t)^p w(t)dt\right)^{\frac{1}{p}} \qquad (1)$$

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if and only if $w \in B_p$.
Moreover, this result remains true for every $p > 0$ (M.J. Carro and J. Soria, [1993]). If $p \leq 1$ then $A = \|w\|_{B_p}^{\frac{1}{p}}$ (M.J. Carro and M. Lorente, [2009]) while if $p > 1$ the best result known up to now is

$$\|w\|_{B_p}^{\frac{1}{p}} \leq A \leq \|w\|_{B_p} \text{ (C.J. Neugebauer, [2010]).}$$

Therefore

$$\|S\|_{L^p_{\text{dec}}(w)\to L^p(w)} \leq \|w\|_{B_p}^{\max(1,\frac{1}{p})},$$

where $Sf(t) = \frac{1}{t}\int_0^t f(s)ds$ is the Hardy operator.

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On the other hand, B. Muckenhoupt proved in 1972 that the inequality (1) holds for every positive function f and $p \geq 1$ if and only if $w \in M_p$, M_p being defined by the condition

$$\left(\int_r^\infty \frac{w(t)}{t^p} dt\right)\left(\int_0^r w(x)^{-\frac{p'}{p}} dx\right)^{\frac{p}{p'}} < \infty.$$

Let us consider a weight $u \in B_p \setminus M_p$ such that

$$\|u\|_{B_p} \lesssim \frac{1}{p-q},$$

for every $p > q$. The symbol $f \lesssim g$ means that $f \leq Cg$, where C is a positive constant. Then, for every $p > q \geq 1$

$$S : L^p_{\mathrm{dec}}(u) \to L^p(u)$$

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Moreover, we can take u such that $u \notin B_q$ and hence S is not bounded on $L^q_{\text{dec}}(u)$.
Can we give some estimate at the endpoint $p = q$?

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Can we give some estimate at the endpoint $p = q$?
This example explains easily the motivation of our work in which the goal is to develop an extrapolation theory for operators acting on decreasing functions, that allow us to obtain end-point estimates as it happens in the classical case.

The weak type of our extrapolation results.

Theorem

Let $p_0 > 1$, $m > 0$ and let u, v, w be three weights such that $W(s)/s$ is a decreasing function. If T is a sublinear operator such that

$$T : L^{p,1}(u)_{dec} \to \Lambda_v^{p,\infty}(w)$$

is bounded for every $1 < p \le p_0$ with constant less than or equal to $\frac{1}{(p-1)^m}$, then

$$T : (L(\log L)^m(\log\log\log L))_{dec}(u) \to \Lambda_v^{1,\infty}(w_m)$$

is bounded, with w_m such that

$$W_m(t) = \frac{W(t)}{(1+\log^+ W(t))^m},$$

and

$$\|f\|_{(L(\log L)^m(\log\log\log L))_{dec}(u)} = \int_0^\infty f(t)\left(1+\log^+\frac{1}{U(t)}\right)^m\left(1+\log^+\log^+\log^+\frac{1}{U(t)}\right)u(t)dt.$$

Given two weights u and v in $\mathbb{R}_+$, the Lorentz space $\Lambda_u^{p,\infty}$ is defined as the set of measurable functions f satisfying

$$\|f\|_{\Lambda_v^{p,\infty}(w)} = \sup_{t>0} W(t)^{1/p} f_v^*(t) < \infty.$$

If the operator T satisfies a stronger condition that in the previous theorem such as that

$$T : L^p_{dec}(u) \to L^p(v)$$

is bounded with the same behavior of the constant, then we can say more, but we need to adapt the techniques from the paper *Convergence a.e. of spherical partial Fourier integrals on weighted spaces for radial functions: endpoint estimates*, M.J. Carro and E. Prestini (2009) to the cone of decreasing functions.

Theorem

Let $p_0 > 1$ *and let* T *be a sublinear operator. If* $T : L^p_{dec}(u) \to L^p(v)$ *is bounded with constant less than or equal to* $\frac{1}{(p-p_0)^m}$ *for* $p > p_0$*, then for every decreasing function* f*,*

$$\sup_{t>0} \frac{\left(\int_0^t [(Tf)^*_v(s)]^{p_0} ds\right)^{\frac{1}{p_0}}}{(1+\log^+ t)^m}$$

$$\lesssim \|f\|_{L^{p_0}(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^{p_0} u(s) ds\right)^{\frac{1}{p_0}}}{U(t)} \left(\log \frac{1}{U(t)}\right)^{m-1} u(t)dt.$$

Example

A generalized Hardy operator.

$$S_v f(t) = \frac{1}{V(t)} \int_0^t f(s)v(s)ds,$$

with v an arbitrary weight. Boundedness properties of these operators have been considered in several papers which can be found in the book *Weighted inequalities of Hardy type* of A. Kufner and L. E. Persson.

If v is a weight satisfying

$$\int_r^\infty \frac{1}{V(x)}dx \lesssim \frac{r}{V(r)},$$

then for every $w \in B_p$ with $p > 0$

$$\|S_v\|_{L^p_{\text{dec}}(w)\to L^p(w)} \lesssim \|w\|_{B_p}^{\max(\frac{1}{p},1)}.$$

As a consequence, for every decreasing function

$$\sup_{t>0} \frac{\left(\int_0^t [(S_v f)(s)]^q u(s) ds\right)^{\frac{1}{q}}}{(1+\log^+ t)} \lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^q u(s) ds\right)^{\frac{1}{q}}}{U(t)} u(t) dt.$$